Amazing Discovery: Professor Bronzin's Option Pricing Models (1908)

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August 2004

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WWZ/Department of Finance, Working Paper No. 8/04

~ Amazing Discovery ~

Professor Bronzin's Option Pricing Models (1908)

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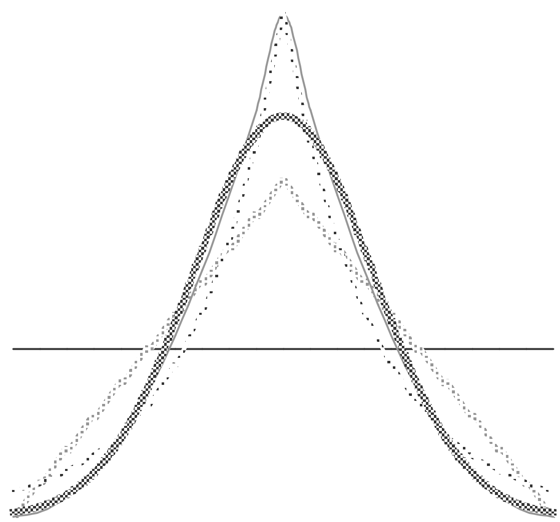

[#] This paper originates in an email sent by the second author wondering whether the first author knew about Bronzin's booklet on option pricing, dating back almost a century and containing formulas which appear rather similar to those developed by Black-Scholes. The scepticism of the first author quickly disappeared after reading Bronzin's manuscript. While the second author continued to work on the history and social use of derivatives, we also had the plan to make the content of Bronzin's booklet accessible to a wider audience. The purpose of this paper is to discuss the major elements of Bronzin's theory and to put it in perspective with modern option pricing models. The content of this paper was first presented in an lecture delivered at the year 2000 Campus of Finance at the Otto Beisheim Graduate School of Management (WHU) in Vallendar, Germany. The paper was written when the first author was a visitor at the University of Chicago. We would like to thank Yvan Lengwiler for his motivation to write this paper, and Christopher L. Culp for his ongoing support and friendship. The detailed and insightful comments by David Rey, Yvan Lengwiler and Stefan Duffner have substantially improved the paper. Comments are welcome at heinz.zimmermann@unibas.ch.

Abstract

The doctoral thesis of Louis Bachelier (1900) is widely considered as the seminal work in option pricing theory. However, only a few years later, 1908, Vinzenz Bronzin, who was a professor of actuarial science at the Accademia di Commercio e Nautica in Trieste, published a booklet (in German) on option pricing as well. While his approach is more pragmatic than Bachelier's, every element of modern option pricing can be found: Risk neutral pricing, no-arbitrage and perfect-hedging pricing conditions, the put-call-parity, and the impact of different distributional assumptions on option values. In particular, he shows how the normal law of error – which is the normal density function – can be used to price options, and how it is related to a binomial stock price distribution. His equation (43) is closer to the Black-Scholes formula than anything published before Black, Scholes, and Merton. He moreover develops a simplified procedure to find analytical solutions for option prices by exploiting a key relationship between their derivatives (with respect to their exercise prices) and the underlying pricing density. Besides of pricing simple calls and puts, he develops formula for chooser options and, more important, repeat-options.

While the book got some attention in the academic literature in the time when it was published (including a not very supportive book review as well), it seems to have been forgotten later. We have just found one modern reference to the book (which is independent of our own research), which is easily verified by a Google search request. It is the purpose of this paper to present the major results of the book, and to highlight its contribution in the light of modern option pricing theory.

Our "discovery" also raises questions beyond the analytics: why did the results of Bachelier, Bronzin, and possibly other's yet to be re-discovered, not get a broader acceptance? Why did their research not find immediate successors, academics that made it a subject of ongoing scientific research? Finding answers to these questions could help us to better understand the cultural background of financial mathematics, and would probably add an interesting chapter to the sociology of science.

Structure of the paper

1. *Introduction*

The doctoral dissertation of the French mathematician Louis Bachelier (1900) is widely considered as the seminal, rigorous work in option pricing theory[1]. While the work remained undiscovered for more than half a century, Bachelier earned his merits after Paul A. Samuelson, based on an inquiry by Leonard J. Savage, discovered the piece, and an English translation of the entire thesis was published in the book of Cootner (1964). Clearly, the merits of Bachelier's work are beyond option pricing; he can be credited for having developed the first mathematical theory of continuous time stochastic processes (the Brownian motion), a few years before Einstein (1905).

This paper is about a different approach to option pricing, taken by an essentially unknown author, Vinzenz Bronzin, only a few years after Bachelier's work was published (1908). This tiny booklet is entitled *Theorie der Prämiengeschäfte* (Theory of Premium Contracts) is written in German and some 80 pages long. While the small booklet got some attention in the academic literature in the time when it was published, it seems to have been forgotten later. For example it was mentioned in a standard banking textbook from Friedrich Leitner in 1920, who was a professor at the Handels-Hochschule of Berlin. Moreover, the book got a short review in the famous *Monatshefte für Mathematik und Physik* in 1910 (Volume 21). But more recent academic mentions are virtually inexistent[2]. Also, only a few biographical details about Bronzin are known to us: he was a professor and later, in the 1920s, the Director of the Accademia di Commercio e Nautica in

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[1] There are numerous references honouring Bachelier's work, e.g. Samuelson (1973), Bernstein (1992), Taqqu (2001), Bouleau (2004), and others.

[2] Except a recent reference from our colleague Yvan Lengwiler (2004), we are aware of only one modern reference on Bronzin's book in a German textbook on option pricing (see Welcker et al. 1988). The authors do not comment on the significance of Bronzin's contribution in the light of modern option pricing theory. A short appreciation of Bronzin's book is also contained in a recent monograph of one of the authors of this paper, Hafner (2002).

Trieste. As a director of this academy he got also a mention in the famous Yearbook of the Scientific World (*Jahrbuch der gelehrten Welt*).

Bronzin's methodological setup is completely different from Bachelier's, at least in terms of the underlying stochastic framework where he takes a much more pragmatic approach. He develops no stochastic process for the underlying asset price and uses no stochastic calculus, but directly makes different assumptions on the share price distribution at maturity and derives a rich set of closed form solutions for the value of options. This simplified procedure is justified insofar as his work is entirely focused on European style contracts (not to be exercised before maturity), so intertemporal issues (e.g. optimal early exercise) are not of premier importance. The methodology of Bronzin is clearly related to his actuarial background.

Bronzin's book contains two major parts. The *first* part is more descriptive and contains a characterization and classification of basic derivative contracts, their profit and loss diagrams, and basic hedging conditions and (arbitrage) relationships. The *second*, and more interesting part is on option pricing and starts with a general valuation framework, which is then applied to a variety of distributions for the price of the underlying security in order to get closed form solutions for calls and puts. Among these distributions is the "error function" which is closely related to the normal distribution. It is interesting to notice that the separation of topics between "distribution-free" and "distribution-related" results is in perfect line with the modern classification of option pricing topics, following Merton (1973).

It would be interesting to know the professional or academic context which lead Bronzin to develop his option pricing theory. Unfortunately, not much is known about this. There is no foreword to the book, no introduction, no information about the author except a short mention as „Professor". But from a book published two years earlier (Bronzin 1906) we know that he was a professor for actuarial theory at the K. K. Handels- und Nautischen Akademie at Triest. Triest was at this time a true melting-pot of people from different nations – James Joyce lived in Triest from 1905 until the beginning of the first world war - and the window of the Donaumonarchie to the Mediterranean Sea. As a center for oversea trading Triest became a European center for insurance. The headquarter of "Generali" is still located in Triest. There are not any references at the end of the book. While the publisher (*Franz Deuticke*) is still in business, the company was not able to provide any information, and even the worldwide web does not provide any meaningful information on Bronzin either[3].

A general difficulty in the attempt to write about Bronzin's book is that the text is written in German, and many of his finance related expressions (which may or may not reflect the commonly used terms at the time being) cannot be translated easily. We therefore have to find English terms as adequate as possible, and add the original German wording in parentheses where it seems to be useful[4]. Moreover we have adapted Bronzin's mathematical notation with only minor changes. In discussing, or extending certain results (particularly in section 5, subsection 5.6), we have tried to make a clear distinction between the results of Bronzin and our own.

We have structured our paper as follows: Section 2 describes the basic terminology as well as the range of derivative contracts analysed in Bronzin's book. Section 3 is about his thoughts on hedging, replication, and arbitrage – although he does not formally use these terms. Section 4 outlines the major elements of his valuation approach, including probabilistic foundations, zero-profit conditions etc. Section 5 discusses the major section of the book, namely the derivation of option prices under alternative specifications of the probability (or pric-

[3] By the time this article is written, a worldwide Google search request on "Vinzenz Bronzin" gives 5 entries: one refers to a website of the authors of Welcker et al. (1988), where the book is quoted in the footnotes, the other four are related to documents released in our own academic environment. Also, searches in electronic archives such as JSTOR did not provide results.

[4] Occasionally, interested readers find important sentences in the full original German wording in footnotes.

ing) density function. This section also shows the close relationship between one of these specifications, the error function, and the Black-Scholes model. Section 6 addresses repeat-options contracts, which play a major role in this book. Section 7 tries to make a final overall assessment of Bronzin's book. It is also natural that we provide many references to Bachelier (1900) throughout the paper.

2. *Classification of contracts, and terminology*

The analysis of Bronzin covers forward contracts as well as options, but his main focus is on the latter. The term "option" does not show up. Instead, his analysis is on "premium contracts" (*Prämiengeschäfte*) which is an old type of option contract used in many European countries up to the seventies, before warrants and traded options became popular[5]. The buyer of a premium contract has the right to step down from a forward contract *at* maturity – not before, so the contract is always European style. The four resulting positions are clearly characterized, analytically as well as in terms of payoff diagrams (pp. 2-7). The buyer of a premium contract acquires either the right to buy (*Wahlkauf*) or to sell (*Wahlverkauf*) the underlying security at maturity, while the seller of the contract has the obligation to sell (*Zwangsverkauf*) or buy (*Zwangskauf*) the underlying. Forward contracts are called "fixed contracts" (*Festgeschäfte*) in Bronzin's terminology (pp. 1-2).

Within the category of option contracts, Bronzin distinguishes between "normal" and "skewed" (*schiefe*) contracts. A normal option contract exhibits an exercise price equal to the forward price, the latter being denoted by B throughout the book. Because of obvious reasons, we will refer to this case as "at-the-money" (ATM) contracts. Skewed call and put contracts exhibit exercise prices which deviate

[5] There is a paper on the French premium market by Courtadon (1982).

by a magnitude M from the forward price. We will denote exercise prices by K in this paper, which implies $K \equiv B + M$.

In addition to these standard (or simple) options, Bronzin analyses two special contracts: choser options (called *Stella-Geschäfte*) where the buyer has the right to determine whether he wants to buy or sell the underlying at maturity[6]; and a special kind of "repeat option" (called *Noch*[7]*-Geschäft*) which adds a (multiple) option component to a forward transaction. The latter contract will be analyzed briefly in Section 6[8].

Throughout the book, Bronzin does not refer to a specific underlying security in his analysis[9], nor to other institutional characteristics of the contracts he analyses. The underlying security is often just called "object" (*Wertobjekt*), and its price is referred to as "market" price.

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[6] They are also shortly addressed by Bachelier; see (p. 53) on *double primes*.

[7] The German word "*noch*" is uncommonly used as a noun here; in common language it is a pronoun and means "another", i.e. an additional one of the same kind, one more.

[8] Also these contracts are shortly analysed by Bachelier; see Section 6 for a direct comparison.

[9] Except in the final numerical example on the second-last page, where he refers to "shares" (*Aktien*).

3. *Hedging, replication, and arbitrage*

3.1 *Hedging and replication*

Two key concepts, "coverage" (*Deckung*) and equivalence (*Äquivalenz*), play an important role in the first part of Bronzin's book (see sections 4 and 5 in chapter I, and section 3 in chapter II). Although the focus of the author is not always clear, this part of the text is nevertheless interesting because, in the light of modern[10] option pricing theory, it is an early notion of perfect hedging and replication of option positions, and the conditions for their general feasibility. Unfortunately, at this stage of analysis, the author does not introduce the concept of arbitrage (or what he later calls "fair pricing"), but discusses the pricing implications rather as "full hedging conditions".

Bronzin defines a "covered" position as a combination of transactions (options *and* forward contracts) which is immune against profits and losses[11]. Two systems of positions are called "equivalent" if one can be derived (*abgeleitet*) from the other, or stated differently, if they provide exactly the same profit and loss for all possible states of the market[12]. From a linguistic point of view, it is interesting to notice that Bronzin explicitly uses the word "derived" in this context.

Bronzin also stresses the relationship between the two concepts: One can always get two systems of equivalent transactions if we take a subset of contracts within a complex of covered transactions and reverse their signs[13]. This basic insight is then followed by a lengthy

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[10] In this paper, „modern" option pricing always refers to the state of the theory after the Black-Scholes-Merton breakthrough.

[11] Original text: „*Wir werden einen Komplex von Geschäften dann als gedeckt betrachten, wenn bei jeder nur denkbaren Marktlage weder Gewinn zu erwarten noch Verlust zu befürchten ist*" (p. 8).

[12] Original text: "*Zwei Systeme von Geschäften nennen wir nämlich dann einander äquivalent, wenn sich das eine aus dem anderen ableiten lässt, in anderen Worten, wenn dieselben bei jeder nur dankbaren Lage des Marktes einen ganz gleichen Gewinn resp. Verlust ergeben*" (p. 10).

[13] Original text: *"... dass wir sofort zwei Systeme äquivalenter Geschäfte erhalten, wenn wir nur in einem Komplexe gedeckter Geschäfte einige derselben mit entgegengesetzten Vorzeichen betrachten*" (p. 10).

characterization of conditions under which combined call and put option positions can be fully "covered" (hedged) – by large systems of equations, which are not easily accessible. He derives the following conclusions[14]:

i) For symmetric i.e. ATM call and put positions (chapter 1, Section 4): The number[15] of long and short options must be equal, *and* the (net or residual[16]) number of long call (put) options must be matched by the same number of forward sales (buys)[17]. Moreover, the call and put price must be equal.

This is a slightly complicated way to state that a long position of calls (puts) plus a short position of puts (calls) produces a synthetic long (short) forward contract. The more interesting part in the statement is the equivalence of option prices

(1)… $C[K = B] = P[K = B]$

(K denotes the exercise price) which is a special case of the well-known put-call parity, mostly ascribed to Stoll (1969), where the exercise price of the options can be arbitrary. For symmetric options, i.e. with exercise prices equal to the forward price, B, the option prices must coincide.

ii) For skewed positions, i.e. calls and puts with arbitrary but equal exercise price (chapter 2, Section 1): The same condi-

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[14] Although not explicit in Bronzin's text, the subsequent hedging conditions refer to options on the same underlying with the same maturity.

[15] Bronzin argues in terms of the „number" of options, but obviously, he assumes equal dollar amounts and equal exposures (which is trivially the case since all options have the same exercise price, B, by assumption).

[16] This specification is not done by Bronzin, but is obvious.

[17] Original text: „*Es müssen ... Wahlgeschäfte in gleicher Anzahl wie Zwangsgeschäfte vorkommen; zu gleicher Zeit müssen aber ... ebenso viele feste Verkäufe desselben Objekts vorgenommen werden, als Wahlkäufe vorhanden sind, oder, was auf dasselbe hinauslaufen muss, ..., ebensoviel feste Käufe abgeschlossen werden, als Wahlkäufe vorhanden sind*" (p. 9).

tions as before must hold, but the equality of call and put price is replaced by the "remarkable" (*bemerkenswert*e) condition

(2)… $P[K = B + M] = C[K = B + M] + M$

which is the put-call-parity, because M measures the "moneyness" of the options. For $M > 0$, the put option is in-the-money, and the equation shows that the put price exceeds the call price by exactly the amount of the moneyness, M. The reverse is true if $M < 0$.

It is important to notice that Bronzin derives this parity relationship as a necessary condition for the feasibility of a perfect hedge[18]. It is apparently obvious for him that a position which is fully hedged against all states of the market cannot exhibit a positive price – but there is no explicit statement of this kind.

Relating his equation to the standard formulation of the put-call parity, it is easy to recognize that the moneyness of the (put) option, M, must just be specified in terms of current dollars as

$$M = Ke^{-r(T-t)} - S_t$$

to get the traditional parity relationship (r is the continuously compounded riskfree rate, $T - t$ time to maturity, and S_t the current stock price[19]. Assuming an interest rate of zero, the forward price collapses with the current spot price, so that the expression simplifies to our case, $M = K - B$.

[18] See e.g. his remark: "*Es müssen überdies zwischen den Prämien der Wahlkäufe und Wahlverkäufe, damit überhaupt eine Deckung möglich ist, die aufgestellten Bedingungen … eingehalten werden …*" (p. 18).

[19] Neglecting dividends, the put-call parity for European style options is $P[K] = C[K] + Ke^{-r(T-t)} - S_t$. Using $M = Ke^{-r(T-t)} - S_t$, which can be solved for $K = (S_t + M)e^{r(T-t)} = B + Me^{r(T-t)}$, the put-call-parity can be expressed as $P[K = B + Me^{r(T-t)}] = C[K = B + Me^{r(T-t)}] + M$.

iii) For calls and puts with various exercise prices (chapter 2, Section 3)[20]: The major insight of Bronzin is what he calls "a strange fact", that a perfect hedge requires that all option series must be covered *separately*, i.e. that there are no hedging effects between different series[21].

It should be noticed that Bronzin does not allow for "delta" hedges (which are not "perfect" in his terminology) because they would require a pricing model, which are not discussed before part II of his text. At the same time, Bonzin recognizes indirect hedging effects between different series through *forward* contracts: Because, as seen in i) and ii), full coverage of individual series requires short or long forward contracts – they may now partially or fully cancel out each other. In a very euphuistic wording, Bronzin characterizes forward contracts as the "powerful intermediaries" (*mächtigen Vermittler*[22]), by which the different option series can be linked to each other.

3.2 Arbitrage

The notion of arbitrage does not show up explicitly in Bronzin's text[23] - but he is close to it. In part I, the put-call-parity is derived as part of the perfect hedging condition for joint put/call positions, without assuming a specific probability distribution for the future market price. It is a distribution-free result. However, no explicit mention is made about arbitrage in the modern sense of the word[24].

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[20] We will subsequently refer to options with different exercise prices (and maturities, which are not considered here) as "series".
[21] Unfortunately, this part of the text (p. 27) is difficult to read, even in German: "... *dass die zu verschiedenen Kursen abgeschloseenen Prämiengeschäfte für sich selbst gedeckte Systeme bilden müssen, ..., wodurch die Unmöglichkeit nachgewiesen wird, Prämiengeschäfte einer einzelnen Gattung durch andere auf Grund verschiedener Kurse abgeschlossener Geschäfte zu decken resp. abzuleiten.*"
[22] From a linguistic point of view it may just be interesting to notice that a different translation of the German „Vermittler" is "arbitrator", which is fairly close to "arbitrage".
[23] The same is also true for Bachelier.
[24] An arbitrage profit is riskfree and does not require a positive amount of capital being invested.

In part II (chapter 1, p. 44 for symmetric contracts, p. 47 for asymmetric contracts) the put-call-parity is again derived, but this time from an explicit *pricing* relationship: Based on a particular price distribution, Bronzin postulates a general valuation principle according to which no profit or loss should be expected for any of the two parties (the buyer and the seller) involved in the transaction when the contract is negotiated. This is a "fair pricing" or "zero expected profit" condition, but no non-arbitrage condition because it refers to expectations, not immediate (risk-free) profits as required by arbitrage. However, Bronzin recognizes that this derivation has a different qualitative nature than in the previous part: the parity relationship has now no longer the character of an artificial condition but emerges from the "incontestable" principle of reciprocity in business transactions[25]. Of course, his remark that the parity only gets its "full justification and importance" at this stage of analysis is not correct, because the derivation in a distribution-free setting is more general. But Bronzin apparently recognizes that deriving the parity by using some kind of "equilibrium" relation adds a new dimension to the pricing of options – although this is not necessary for the put-call-parity itself, but for the other pricing relationships he is about to derive.

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[25] This is a rather free translation. The orginal text reads as: "... *sondern dem unanfechtbaren Prinzip der Gleichheit von Leistung und Gegenleistung entsprungen ist*" (p. 48).

4. *The general valuation framework*

4.1 *Spot price, expected price, and forward price*

Bronzin recognizes that his analysis in part I of his booklet leaves open the fundamental question about the appropriate (*rechtmässig*) size of the option premiums. He also recognizes that further assumptions and tools[26] are necessary to achieve this goal: probabilistic assumptions about the market[27], and a rule to translate expected profits and losses from the contracts to current values.

The market model which Bronzin has in mind can be characterized as a driftless random walk. In this respect, the approach is virtually identical to Bachelier (1900).

- *Random walk*. When discussing the possible specification of the probability density function of the underlying market price (p. 56), he finds himself in substantial difficulties: He argues that he does not know any general criteria to characterize the random (*regellos*) market movements for the various underlyings analytically[28]. Instead, he proposes to statistically estimate possible distributions (see Section 5).

- *Spot and forward price*. The starting point of Bronzin's probabilistic market model is the forward price B. He assumes that this price is "naturally" close or even identical to the current spot price[29]. Since there is no mention about interest rates, the time

[26] He notices that the tools which are required for this task are beyond elementary mathematics – only the application of probability and integral calculus is able to shed light on this important question (p. 39).
[27] It is interesting to notice that his focus is from the beginning on the variability (volatility) and the current state of the market (*Marktschwankungen*), not the trend.
[28] Original text: "*Allgemeine Anhaltspunkte, um die regellosen Schwankungen der Marktlage bei den verschiedenen Wertobjekten rechnerisch verfolgen zu können, gehen uns vollständig ab*" (p. 56).
[29] Original text: „*... zum Kurse B, welcher natürlicherweise mit dem Tageskurse nahe oder vollkommen übereinstimmen wird....*" (p. 1).

value of money, or discounting anywhere in the book, this also implies that he assumes an efficient market.

- *Price expectation*. He repeatedly argues that the forward price is the most likely among all possible future market prices (p. 56, p. 74, p. 80), i.e. the forward price is an unbiased predictor of the future spot price. Otherwise, he argues, one could not imagine sales and purchases (i.e. opposite transactions) with equal chances if strong reasons would exist leading people to ultimately predict either a rising or falling market price with higher probability[30]. Thus, the forward price is regarded as the most advantageous price for both parties in a forward transaction[31]. A slightly different reasoning is used when discussing the payoff diagram of a forward contract, where he states that the forward price B must be such that the two "triangle parts" to the left and the right of B, i.e. to the profit and loss of the contract, must be "equivalent" because otherwise, selling or buying on spot should be more profitable[32]. This does not necessarily imply an unbiased forward price, although there is little doubt that he wants to claim this.

While the issue of price expectations seems to be important for Bronzin, it is not relevant for the development of his model. The important point is that the mean of the price distribution is based on *observable* market price (spot or forward price), not price expectation or other preference-based measures[33]. Whether the forward price

[30] Original text: "*Es könnten ja sonst nicht Käufe und Verkäufe, d.h. entgegengesetzte Geschäfte, mit gleichen Chancen abgeschlossen gedacht werden, wenn triftige Gründe da wären, die mit aller Entschiedenheit entweder das Steigen oder das Fallen des Kurses mit grösserer Wahrscheinlichkeit voraussehen liessen*" (p. 74).

[31] On (p. 56), the reasoning for this insight is justified by the fact, that the call and put prices coincide if the exercise price is equal to the forward price.

[32] Original text: "*Es braucht kaum der Erwähnung, dass die dreieckigen Diagrammteile rechts und links von B als äquivalent anzunehmen sind, da sonst entweder der Kauf oder der Verkauf von Haus aus vorteilhafter sein sollte*" (p.1). The wording "von Haus aus" is no longer known in the German language, but it obviously means a spot transaction.

[33] The same is true for Bachelier's analysis. In contrast to Bronzin, he does not argue with the forward price, but he apparently assumes that the price at which a forward contract (*opération ferme*) is executed is

matches the expected future price or not does not affect the subsequent results. This would be relevant if statement on risk premiums or risk preferences should be made, which is not the intention of the author. Instead, his focus is on *consistent* (or in his wording, fair) pricing relationships between spot, forward, and option contracts – exactly like in the modern setting of arbitrage-based valuation models.

4.2 An asymmetric probability density of the underlying price

As noted before, Bronzin understands the forward price as the cutting edge for modeling the ups and downs of the underlying market price. He consequently characterizes the random behavior of the market price by its deviation from the forward price, $\tilde{x} \equiv \tilde{S}_T - B$, where $\tilde{S}_T$ is the stock price at maturity (which is however never focused throughout the text).

Interestingly, in the general setting of his model (chapter 1), Bronzin allows for different probability distributions for the upstates and the downstates of the market: The probability that the market price exceeds B and takes a value between $B + x$ and $B + x + dx$ is characterized by $f(x)dx$, while the probability that the underlying price is below B and takes values between $B - x$ and $B - x - dx$ is $f_1(x)dx$. Since many of the assumed probability distributions are bounded over a finite range, lower and upper boundaries ω_1 and ω are used. Notice that the cumulative probabilities of the two segments can be different from 50%. The expected profit of a specific instrument with payoff function G in an upmarket is thus given by $\int_0^{\omega} Gf(x)dx$, the expected profit in a in downmarket is $\int_0^{\omega_1} Gf_1(x)dx$, respectively.

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equal to the current spot price (see his characterization on p. 26; notice that his x is the deviation of the stock price at expiration from the current value).

The probability that $\tilde{x}$ exceeds a certain threshold value, say x^*, at option maturity plays a key role in the subsequent analysis. We could write this as $F(x^*) \equiv \int_{x^*}^{\omega} f(x) dx$. Unfortunately, Bronzin (equation 5, p. 41 throughout the rest of the book) uses a rather misleading notation by not distinguishing between the random variable $\tilde{x}$ and the threshold value x^*. In order to minimize the discrepancies with his and our text, we adapt this sloppy notation and write

$$(3)\ldots \quad F(x) \equiv \int_{x}^{\omega} f(x) dx$$

Apparently, the sign of $\frac{\partial F(x)}{\partial x}$ is negative.

4.3 The expected value of a contract is zero – an analogy with the risk-neutral valuation approach

Bronzin (pp. 41/42) then states the important valuation principle that at contract settlement, no profit or loss should be expected[34] for any of the two parties (the buyer and the seller) involved in the transaction. For this purpose, the conditions of each transaction must be determined in a way that the sum of expected profits of both parties (taking losses as negative profits) is zero[35]. Bronzin calls this the "fair pricing" condition (*Bedingung der Rechtmässigkeit*). Obviously, it is a zero profit condition assuming that there is no time value of money and no compensation for risk. It is the same assumption Bachelier makes to justify the martingale assumption of stock prices[36].

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[34] It is important to notice that the statement, in the literal sense, is about expected, not current (riskless), profits. It is therefore not a no-arbitrage condition. Original text: "*... dass im Moment des Abschlusses eines jeden Geschäfts beide Kontrahenten mit ganz gleichen Chancen dastehen, so dass für keinen derselben* IM VORAUS *weder Gewinn noch Verlust anzunehmen ist*" (p. 42).

[35] Original text: "*wir stellen uns also jedes Geschäft unter solchen Bedingungen abgeschlossen vor, ... dass der gesamte Hoffnungswert des Gewinns für beide Kontrahenten der Null gleichkommen müsse*" (p. 42).

[36] For example: "*L'espérance mathématique du spéculateur est nulle*" (p. 18); "*Il semble que le marché, c'est-à-dire l'ensemble des spéculateurs, ne doit croire à un instant donné ni à la hausse, ni à la baisse,*

One may be tempted to argue that this is appropriate for short time intervals, but not for time periods relevant for most derivative contracts. But, as discussed in Section 3.1, the assumption of risk neutrality is nevertheless applicable because the authors follow a *relative* pricing approach where risk preferences and expectations do not matter. Thus, one just has to re-interpret their statistical density functions as *risk neutral* densities in the modern sense. It was the seminal paper by Harrison/ Kreps (1979) demonstrating that a martingale under an equivalent (i.e. risk neutral) probability measure is a sufficient and necessary condition for the absence of arbitrage profits. Therefore, the martingale stories (no speculative profit, number of buyers equal to the number of sellers, etc.) of Bronzin and Bachelier are nevertheless essential for their valuation models – but not in the statistical sense they have in mind, but with respect to a modified density: And indeed, the densities of both authors qualify as risk-neutral densities because their means do not depend on preferences or expectations. Bachelier's pricing density is centered around the current stock price (or the absolute returns around zero); Bronzin's pricing density is centered around the forward price (which is "close to or even matches" the current market price).

Moreover, as discussed in Section 4.4, both authors show the analytical link between probability densities and option prices (Bronzin's equation 17, and Bachelier on p. 51), which demonstrates that they were fully aware that probabilities are directly linked to observed market prices in their valuation framework – but not the statistical, but risk neutral probabilities.

puisque, pour chaque cours coté, il y a autant d'acheteurs que de vendeurs" (pp. 31-32); "*L'espérance mathématique de l'acheteur de prime est nulle*" (p. 33).

It follows that the statistical probability densities of Bronzin (and Bachelier) can be easily re-interpreted as (deflated[37]) risk neutral pricing densities in the modern sense. Simple mathematical expectation can then be taken to compute prices.

The pricing approach of Bronzin is illustrated with a simple ATM call: The expected profit if the market exceeds the forward price B is $G=\int_0^{\omega}(x-P)f(x)dx$, where P is the price of the call option. Notice that because there is no time value of money, the option premiums can be added and subtracted from the terminal payoff. The expected loss in the down market is respectively $V=\int_0^{\omega_1}Pf_1(x)dx$, and the zero profit condition implies

$$G-V=\int_0^{\omega}(x-P)f(x)dx-\int_0^{\omega_1}Pf_1(x)dx=0$$

which can be solved for the option price

$$P=\int_0^{\omega}xf(x)dx\,.$$

For out-of-the-money calls ($X=B+M$), the profit and loss function is defined over 4 consecutive market price intervals bounded by $[\omega_1;B,B+M;B+M+P_1;\omega]$, and thus generalizes to

$$-\underbrace{\int_0^{\omega_1}P_1f_1(x)dx}_{V_3}-\underbrace{\int_0^{M}P_1f(x)dx}_{V_2}-\underbrace{\int_M^{M+P_1}(M+P_1-x)f(x)dx}_{V_1}+\underbrace{\int_{M+P_1}^{\omega}(x-M-P_1)f(x)dx}_{G}=0$$

---

[37] Deflated by the risk-free rate of interest; because Bachelier and Bronzin implicitly assume that interest rates are zero, discounting does not matter in either case. See also section 4.5.

where 3 loss components must be taken into account. This yields after some manipulations

(4a)… $$P_1 = \int_M^{\omega} (x - M) f(x) dx$$

The price of the equivalent in-the-money put option is derived as

(4b)… $$P_2 = \int_0^{\omega_1} (M + x) f_1(x) dx + \int_0^{M} (M - x) f(x) dx$$

which after some manipulations (p. 47) leads to the put-call-parity

(4c)… $$P_2 = P_1 + M \, .$$

as discussed earlier.

4.4 Substituting probabilities by prices

In our opinion, the most amazing part of Bronzin's booklet is in Section 8 of the first chapter in part II, where he relates the probability function $f(x)$ to option prices. This was explicitly done in an unpublished paper by Black (1974) [38], and a few years later by Breeden/Litzenberger (1978). By referring to the rules of differentiation with respect to boundaries of integrals, and expressions within the integral (generally known as Leibnitz rules), he derives the "remarkable" expression

(5)… $$\frac{\partial P_1}{\partial M} = -\int_M^{\omega} f(x) dx = -F(M)$$ (equation 16, p. 50).

[38] Many years ago, William Margrabe made me aware of this paper. Not many people seem to know this tiny piece; e.g. it is also missing in the Merton and Scholes *Journal of Finance* tribute after Fischer Black's death, where all his papers and publications are listed.

Remember that $F(M)$ is the probability that the stock price exceeds the exercise price at maturity, i.e. that the options gets exercised. Equation (5) thus postulates that the negative of the exercise probability is equal to the first derivative of the option price with respect to the exercise price (respectively, M).

The important point is that this expression is much easier to solve for P_1 than in the standard valuation approach: Based on (5), the option price can be computed by the indefinite integral

(6)… $$P_1 = -\int F(M)dM + c$$ (equation 19, p. 51)

where c is a constant which is not difficult to compute (it will be zero or negligible in most cases). Equation (6) is a powerful result: Option prices can be computed by integrating $F(M)$ over M. Depending on the functional form of $f(x)$, this could drastically simplify getting option values. Thus, knowing (or determining) the function $F(x = M)$ showing the exercise probabilities as a function of M, is the key element in determining option values under this approach.

From there, it is straightforward to show that the second derivative

(7)… $$\frac{\partial^2 P_1}{\partial M^2} = f(M)$$ (equation 17, p. 51)

directly gives the value of the (probability density) function at $x = M$[39]. As Breeden/ Litzenberger (1978) have shown, this derivative multiplied by the increment dM can be interpreted as the implicit state price[40] in the limit of a continuous state space. Absence of arbitrage requires that state prices are strictly positive, which implies

[39] Interestingly, Bachelier (1900) on p. 51 also shows this expression, but without motivation, comments, or potential use.
[40] For a discrete distribution of states, the state price (also called Arrow-Debreu price) is the current price of a claim, which entitles its owner to receive a dollar in one specific future state, and nothing otherwise.

$\frac{\partial^2 P_1}{\partial M^2} > 0$, i.e. option prices must be convex with respect to exercise prices. If this is condition is not satisfied, a butterfly spread[41] would generate an arbitrage profit. Bronzin also shows that equation (7) can be applied without adjustments to put options.

It is apparent from Bronzin's equations (16), (17) and (19), that he was aware that information on the unknown function $f(x)$ is impounded in observed (or theoretical) option prices, and just need to be extracted. It establishes $f(x)$ as a *pricing* function (or density), or to put it more directly: they demonstrate the key relationships between security prices and probability densities.

In complete markets, probabilities can be substituted by (observed or observable) prices; this needs some comments later in this Section. This has far reaching consequences, both for theoretical as well as applied (empirical) research.

Implications for empirical work

Although the main interest of Bronzin are the analytical consequences of this finding, he is well aware of the empirical implications. He claims the difficulties in specifying the function $f(x)$ on a priori grounds (p. 56)[42] and suggests to fit the function $F(x)$ (see equation 3) with empirical data[43]: For different predetermined values of x, com-

[41] This is a strategy where three options contracts (on the same underlying) with different exercise prices are bought and sold. If the exercise prices are $K - \Delta K$, K and $K + \Delta K$, the strategy is to sell two contracts at K and buy one contract at $K - \Delta K$ and one at $K + \Delta K$. Any non-convexities in the corresponding option prices $P_1(K - \Delta K)$, $P_1(K)$ and $P_1(K + \Delta K)$ can be exploited by this strategy.

[42] Original text: „*Was nun die Form der Funktion $f(x)$ selbst anlangt, so stossen wir auf sehr grosse Schwierigkeiten. Allgemeine Anhaltspunkte, um die regellosen Schwankungen der Marktlage bei den verschiedenen Wertobjekten rechnerisch verfolgen zu können, gehen uns vollständig ab*“ (p. 56).

[43] $F(x)$ denotes the probability that the market price exceeds a predetermined value x.

pute the relative frequency $\frac{g}{m}$ by which the market price exceeded x in the past:

$$F(x_j) = \int_{x_j}^{\omega} f(x_j)dx_j = \frac{g_j}{m_j} \; \forall j$$

He then suggests the functional form of $F(x = M)$ could then be determined by running a least-square regression of the empirical $F(x_1),...,F(x_n)$ values on $x_1,...,x_n$. He claims, quite correctly, that this procedure generates for every possible underlying it's specific $F(x)$ function, which would be very handy, and relating the result to $\frac{\partial P_1}{\partial M} = -F(M)$ could answer any question in a simple and reliable way … (p. 57). However, being a mathematician, he then says that he does not want to do this troublesome job, but is satisfied with specific functional specifications of $f(x)$. This will be discussed in section 5.

Analytical implications

The analytical implications of equations (5) and (6) are of key interest to Bronzin, and we therefore provide a brief illustration using the "triangle distribution" which he uses later in his analysis. $f(x)$ is specified as a linear function $f(x) = a + bx$, defined over the interval $[0;+\omega]$; and respectively $f_1(x) = a + b|x|$ if x is in the negative range $[-\omega;0]$. For $f(\omega) = f_1(\omega) = 0$ to hold, the parameters must be specified as $a = \frac{1}{\omega}$, $b = -\frac{1}{\omega^2}$, which implies $f(x) = \frac{\omega - x}{\omega^2}$.

The standard pricing approach requires the solution of the integral

$$P_1 = \int_M^{\omega} (x - M) f(x) dx = \int_M^{\omega} (x - M) \frac{\omega - x}{\omega^2} dx$$

which is a quite complicated task (see p. 66). In contrast, the procedure suggested by Bronzin is much simpler:

- Compute $F(M)$, i.e. the probability that $\tilde{x}$ exceeds $x = M$. This given by $\frac{(\omega - M)^2}{2\omega^2}$.
- Solve $\frac{\partial P_1}{\partial M} = -F(M) = -\frac{(\omega - M)^2}{2\omega^2}$ for P_1, which is given by the integral $P_1 = -\int F(M)dM + c = -\int \frac{(\omega - M)^2}{2\omega^2} dM + c$. The solution is $P_1 = \frac{(\omega - M)^3}{6\omega^2}$. Notice that the constant is zero because $P_1(M = \omega) = 0$ (see p. 62).

A graphical illustration is provided by **Figures 1a-1c**. We assume $\omega = 10$ and an exercise price of $M = 5$. The resulting (call) option price is 0.208.

Summing up

The major task in pricing options and other derivatives is to find an appropriate pricing function $f(x)$ which translates future random payoffs into current prices. In Bronzin's own perspective, $f(x)$ is a standard probability density function. However, the mean of his density is not an ordinary, unspecified or subjective expected value, but a market price which can be observed – namely the forward price of the security. His pricing density can thus be regarded as a risk-neutral pricing function – which does not necessarily provide the "correct" statistical probabilities, but prices options in a consistent way with the underlying resp. the forward contract. Based on his equations (16), (17) and (19), Bronzin suggests *three* ways to specify the pricing function $f(x)$:

- Estimate volatilities and probabilities, and fit $F(x)$ by least squares (the derivative $f(x)$ can then be derived).
- Try alternative functional specifications (see Section 5).
- Compute the second derivative $\frac{\partial^2 P_1}{\partial M^2} = f(M)$ for alternative $M = x$ from existing market prices; as shown in equation (7).

In contrast to the first and second alternative, Bronzin does not put emphasis on the third[44]. Of course, it is not very obvious to derive option prices by relying on (these) option prices. However, in the light of modern state pricing theory, it would be useful to comment on it anyway, simply because modern financial markets (and advanced econometric methods) offer many opportunities to estimate functions of this kind.

Let us briefly examine the economic implications of Bronzin's discussion about the specification of $f(x)$. As mentioned earlier, from the perspective of state preference theory, the derivative $\frac{\partial^2 P_1}{\partial M^2} dM = f(M) dM$ can be interpreted as state price in state $x = M$ in the limit of a continuous state space[45]. A necessary and sufficient condition for markets to be complete is that the distribution of state prices is uniquely defined over the whole range of states. Thus, the key question is whether this distribution can be directly inferred from market data (e.g. by synthetically creating state securities); this depends on the completeness of the underlying security market[46].

An immediate interpretation of equation (7) would be that Bronzin recognizes that the distribution of $f(x)$ can be recovered from option

[44] Except from mentioning that it is a differential equation for P_1 without integrals.

[45] The terms "state price density" and "risk neutral density" can be used synonymously here, because interest rates are zero. See the appendix to this section for a short characterization.

[46] An excellent discussion about completeness and near-completeness of markets in the context of options can be found in Lengwiler (2004), pp. 118-120.

prices (or other securities from which options can be replicated). Market completeness would require, however, that there is a continuum of exercise prices. Since in reality only an (extremely) finite number of exercise prices is available for most securities, state prices can be, at best, approximated by

$$\left\{\frac{P_1(M+\Delta M)-P_1(M)}{\Delta M}\right\}-\left\{\frac{P_1(M)-P_1(M-\Delta M)}{\Delta M}\right\}=f(M)\Delta M, \quad \forall M=x.$$

Thus, equation (7) is an interesting theoretical insight for Bronzin, but should not be interpreted as recognition of market completeness. He repeatedly stresses that the knowledge on $f(x)$ is extremely limited, and his suggestion how to fit the empirical estimates of $F(x)$ on x is a clear illustration of this point. In contrast to essentially all other early authors, he is very explicit on the "model risk" in option pricing (without using this word) but trying out different functional specifications of $f(x)$ and comparing the resulting option values. The distributions mainly differ with respect to the modeling of "extreme" events.

Appendix

This appendix reviews a useful relationship. Define:

$f(x)dx$	state (or Arrow-Debreu) price
$f(x)$	state price density
$f(x)=e^{-r}\hat{p}(x)$	$\hat{p}$ is the risk-neutral density function, i.e. the state price density after risk-free discounting

The relationship between the statistical and the risk-neutral probability functions requires an additional definition:

$m(x)$	is the “stochastic deflator”, which is state price per probability unit; it shows the pure state preference of the individuals

$$m(x) = \frac{f(x)dx}{p(x)dx} = \frac{e^{-r}\hat{p}(x)dx}{p(x)dx} = \frac{price}{prob}$$

A stochastic deflator is a state-dependent (random) variable satisfying

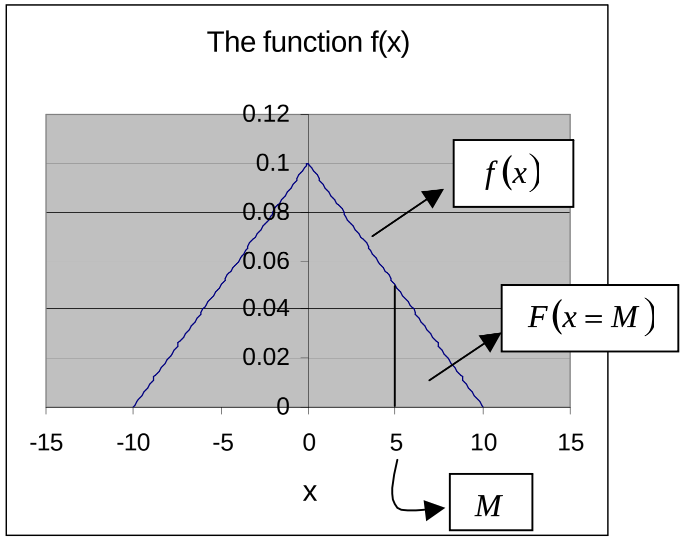
$$P_1 = E[m(x)\Omega(x)] = \int_s m(s)\Omega(s)p(s)ds$$

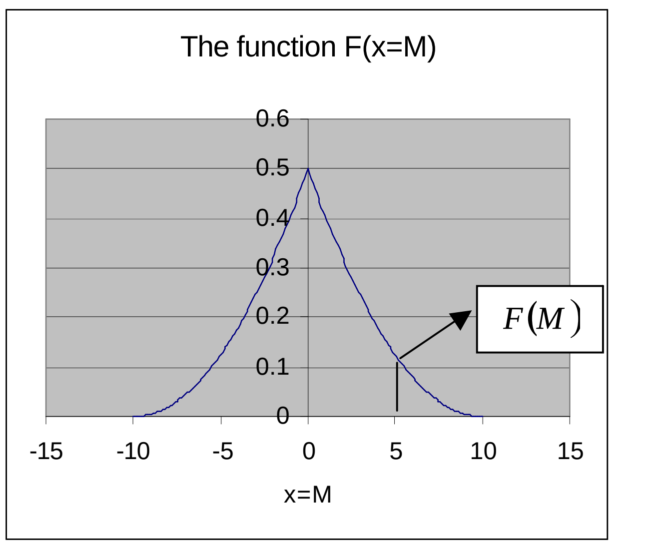
where P_1 is the current price of a security with future random payoff $\Omega(x)$. It then follows that the relationship between $p(x)$ and $\hat{p}(x)$ is given by

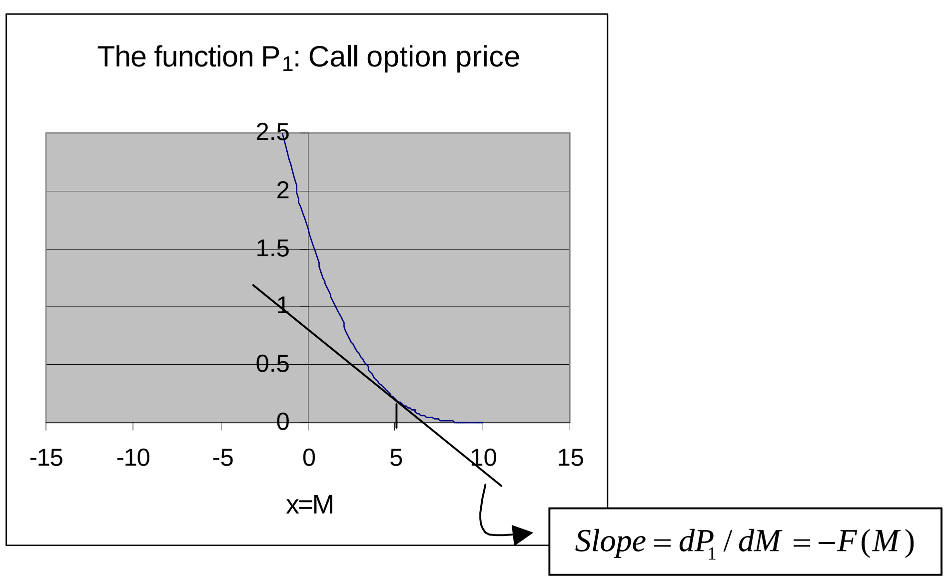
$$\hat{p}(x)dx = m(x)e^{r}p(x)dx \, .$$

In a complete market, $\hat{p}(x)$ and $m(x)$ are unique. This is no longer true in an incomplete market.

Figures 1a-1c The Bronzin approach to option pricing – or: three ways to represent the exercise probability $F(M)$ of an option: Illustration with the triangle pricing density







5. *Option pricing with specific functional or distributional assumptions*

5.0 Pre-Remarks

The specification of the pricing density $f(x)$ and the derivation of closed form solutions for option prices is the objective of the 2nd chapter in part II. Bronzin discusses six different functional specifications of $f(x)$ and the implied shape of the density for a given range of x. From a probabilistic point of view, this part of the book seems to be slightly outdated, because the first four "distributions" lack any obvious stochastic foundation. The function $f(x)$ seems to be specified rather ad-hoc, just to produce simple probability shapes for the price deviations from the forward price: a rectangular distribution, a triangular distribution, a parabolic distribution, and an exponential distribution.

This impression particularly emerges if Bachelier's thesis is taken as benchmark, where major attention is given to the modeling of the probability law governing the dynamics of the underlying asset value. This was an extraordinary achievement on its own. In order to be fair about Bronzin's approach, one should be aware of the state of probability theory at the beginning of the last century. As Bernard Bru mentioned in his interview with Murad Taqqu (see Taqqu 2001, p. 5), "probability did not start to gain recognition in France until the 1930's. This was also the case in Germany".

However, the fifth and sixth specification of $f(x)$ are the (normal) law of error (*Fehlergesetz*) and the Bernoulli theorem, or in modern terminology, the normal and binomial distributions. This enables a direct comparison with the Bachelier and the Black-Scholes and Merton models. This implies that Bronzin was familiar with basic statistical models. Moreover, even the four "ad-hoc" models are special cases of more general family of error laws, called "Pearson laws"[47]. Moreover, the *triangular* distribution can be understood as the sum of two random variables with a rectangular distribution; and the *parabolic* distribution as the sum of three random variables with a rectangular distribution; see Jeffreys (1939/ 1961, pp. 101-103) for discussing the convergence of sums of error distributions. This shows that the rectangular distribution, despite its unrealistic shape for securities prices, is not an unreasonable choice to start with.

Based on these arguments, Bronzin's specifications of $f(x)$ are not so arbitrary as they may appear at first sight. The discussion in the next Sections will moreover show that analyzing option prices in this simple setting has great didactical benefits.

For the subsequent discussion it is useful to recall that x denotes the market price of the underlying asset at maturity *minus the forward price*. Bronzin now makes the simplifying assumption that functions $f(x)$ and $f_1(x)$ are symmetric around B, i.e. that $f(x) = f_1(x)$. This implies[48] $\omega = \omega_1$, and consequently, $\int_0^{\omega} f(x)dx = 0.5$ (p. 55). This assumption makes the expected market price equal to the forward price; as discussed earlier, Bronzin considers this a straightforward (*a priori einleuchtend*, p. 56) economic assumption. At the same time, he is entirely aware that a symmetric probability density is not consistent with the limited liability nature of the underlying "objects": while price increases are potentially unbounded, prices cannot fall below zero[49]. However, he plays this argument down by saying that these (extreme) cases are fairly unlikely, and price variations can be re-

[47] See e.g. Jeffreys (1939, 1961), pp. 74-78. This book was very helpful for us in understanding the terminology on the normal distribution, called the normal law of error, as used at the beginning of the past century.

[48] Notice that $\int_0^{\omega} xf(x)dx = \int_0^{\omega_1} xf_1(x)dx$ must hold.

[49] Original text: „... *es könnte ja eine Kurserhöhung in unbeschränktem Masse stattfinden, während offenbar eine Kurserniedrigung höchstens bis zur Wertlosigkeit des Objekts vor sich gehen kann*" (p. 56).

garded as more or less uniform (*regelmässige*) and generally not substantial (*nicht erhebliche*) oscillations around B. Based on this reasoning, he seems to be very confident about the results being derived from this assumption…[50]

5.1 A constant (Rectangular distribution)

In a first step, it is assumed that $f(x)$ is a constant within $[-\omega;+\omega]$. This implies that the function must be zero at the boundaries of the integral, $f(\omega)=0$, which implies the simple functional specification

$$f(x)=\frac{1}{2\omega}$$

for the pricing density. Based on this function, we are able to derive the cumulative density function $F(x)$. Evaluated at $x=M$, this function which can be understood as the negative of the first derivative of the option price with respect to the exercise price at $B+M$, i.e. $F(M)=-\frac{\partial P_1}{\partial M}$. Simply integrating this expression over M gives the option price (plus a constant). Because this valuation procedure is similar for all specifications discussed in the subsequent Sections, we will adapt a standardized way to present the results. The major elements and results of the valuation procedure are presented in Tables; the second column displays the important formulae, the third column contains complimentary equations (assumptions etc.)[51].

The results for this distribution are in **Table 1**. Interpreting ω as volatility of the underlying, the formula neatly separates the impact of intrinsic value M and volatility on option price. As done for other

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[50] Original text: „ .... *so darf man die gemachte Voraussetzung getrost akzeptieren und ihren Resultaten mit Zuversicht entgegensehen*" (p. 56).
[51] If not mentioned otherwise, the results in the Tables are those derived by Bronzin, while the interpretation in the text is our's.

specifications, the relationship between the ATM call price P and general call price P_1 is given by

$$P_1 = \left(1 - \frac{M}{4P}\right)^2 P$$

Also, the symmetry between put and call prices with respect to the forward price is easily recognized. Of course, the distribution is unrealistic for most practical applications, but the pedagogical merits are straightforward.

Table 1 – The function $f(x)$ is a constant (rectangular distribution)

Function $f(x)$	$f(x) = a$	$f(\omega) = 0$
Density $f(x)$	$f(x) = \frac{1}{2\omega}$	
Exercise probability $F(x = M)$	$F(M) = \frac{\omega - M}{2\omega}$	
Pricing kernel	$\frac{\partial P_1}{\partial M} = -F(M) = -\frac{\omega - M}{2\omega}$	
Call	$P_1 = \frac{(\omega - M)^2}{4\omega}$	
ATM Call/Put	$P = \frac{\omega}{4}$	
Put	$P_2 = \frac{(\omega + M)^2}{4\omega}$	

5.2 A linear function (Triangular distribution)

Next, the function $f(x)$ is assumed being linear within the subintervals $[-\omega;0]$ and $[0;+\omega]$. The implied density function is then a symmetric triangle with its vertex equal to $1/\omega$ at the forward price; see **Figure 2a**. The rest of the pricing equations is displayed in **Table 2**. Assuming the same boundaries ω as in the previous Section[52], it is interesting to notice that the ATM option prices decrease from one fourth of ω (as for the uniform distribution) to one sixth. This nicely shows the impact of shifting part of the probability mass (i.e. one eighth on each side of the distribution) from the "tails" to the center of the distribution, or the reverse. To put it differently, the "riskier" uniform density implies an ATM option price which is $\omega/4 \div \omega/6 = 1.5$ times, or respectively 50%, higher than the price implied by the triangular distribution – although only 25% of the probability mass is shifted from the tails to the center.

Again, as in the previous Section, the non-ATM call price can be easily decomposed to an intrinsic and volatility part.

Table 2 – The function $f(x)$ is linear (triangular distribution)

Function $f(x)$	$f(x) = a + bx$	$f(\omega) = 0$
Density $f(x)$	$f(x) = \frac{\omega - x}{\omega^2}$	$a = \frac{1}{\omega}$, $b = \frac{-1}{\omega^2}$
Exercise probability $F(x = M)$	$F(M) = \frac{(\omega - M)^2}{2\omega^2}$	
Pricing kernel	$\frac{\partial P_1}{\partial M} = -F(M) = -\frac{(\omega - M)^2}{2\omega^2}$	$P_1(M = \omega) = 0 \Rightarrow c = 0$
Call	$P_1 = \frac{(\omega - M)^3}{6\omega^2}$	
ATM Call/Put	$P = \frac{\omega}{6}$	
Relation between ATM Call and general Call	$P_1 = \left(1 - \frac{M}{6P}\right)^3 P$	

[52] This does not keep the standard deviation of the distribution the same, of course.

5.3 A quadratic function (Parabolic distribution)

In the next step, Bronzin assumes a quadratic function for $f(x)$ within the interval $[-\omega;0]$ and $[0;+\omega]$. Notice the conditions under which the parameters $\{a,b,c\}$ are derived. Note that $f'(x=\omega)=0$ ensures that the function has its minimum at $x=\omega$ where it asymptotically approaches the abscissa. Compared to the triangular distribution discussed before, the probability of reaching ω is (again) smaller; see **Figure 2b**. Bronzin suggests to use this distribution for modeling extreme values with small probabilities by setting ω sufficiently large (p. 67). Nevertheless, we now assume that ω is the same as in the previous two Sections in order to facilitate comparisons.

Since extreme value have again become less likely compared to the triangular distribution, it is not surprising that the value of ATM options is again lower, i.e. it decreases from one sixth of ω to one eighth. The other results are similar and need no further comment.

Table 3 – The function $f(x)$ is quadratic (parabolic distribution)

Function $f(x)$	$f(x)=a+bx+cx^2$	$f(\omega)=0$, $f'(x=\omega)=0$
Density $f(x)$	$f(x)=\frac{3(\omega-x)^2}{2\omega^3}$	$a=\frac{3}{2\omega}$, $b=\frac{-3}{\omega^2}$, $c=\frac{3}{2\omega^3}$
Exercise probability $F(x=M)$	$F(M)=\frac{(\omega-M)^3}{2\omega^3}$	
Pricing kernel	$\frac{\partial P_1}{\partial M}=-F(M)=-\frac{(\omega-M)^3}{2\omega^3}$	$P_1(M=\omega)=0 \Rightarrow c=0$
Call	$P_1=\frac{(8\omega-M)^4}{8\omega^3}$	
ATM Call/Put	$P=\frac{\omega}{8}$	
Relation between ATM Call and general Call	$P_1=\left(1-\frac{M}{8P}\right)^4 P$	

5.4 An exponential function (Negative exponential distribution)

Finally, an exponential distribution is assumed for $f(x)$; in contrast to the functions assumed before, the range of x over which the function is defined, needs no arbitrary restriction. The function asymptotically converges to zero for large x; **see Figure 2c**. The range of x values is unbounded, and rare events with small probabilities can even be handled much easier by this functional specification. The parameter k determines the variability of x – a bigger k reduces the variability. As shown in the next Section, the standard deviation (volatility) of the distribution is given by $\sigma = 1/2k$. Then the price of ATM option is straight *half* the volatility! Again, the general option prices separate the impact of the volatility and moneyness in an extremely nice way.

The comparison with the option price derived from the previous distribution (quadratic) is not straightforward. First, we should know the probability by which the exponential distribution exceeds the maximum value of the parabolic distribution ω; this is given by the function $F(x=\omega)=\frac{e^{-2k\omega}}{2}$ (see Bronzin p. 70, equation 30). We then calibrate k such that the exponential function is identical to the quadratic at $x=0$. The quadratic function is $f_q(x=0)=\frac{3}{2\omega}$, and setting it equal to the exponential at $x=0$, $f_{\exp}(x=0)=k$, we get $k=\frac{3}{2\omega}$. The probability that realizations from the exponential density exceed the maximum of the parabolic, ω, is therefore

$$F(x=\omega)=\frac{e^{-2\frac{3}{2\omega}\omega}}{2}=\frac{e^{-3}}{2}=0.02489$$

which is approx. 2.5%, or on a two sided basis, 5%. So it is easy to find how the "extra" risk is rewarded. The ATM option price under our calibration for k is

$$P\left(k=\frac{3}{2\omega}\right)=\frac{1}{4k}=\frac{1}{4\times\frac{3}{2\omega}}=\frac{1}{\frac{6}{\omega}}=\frac{\omega}{6}$$

which exceeds the respective option price from the parabolic distribution by $\frac{\omega/6}{\omega/8}-1=\frac{1}{3}$, i.e. one third.

Table 4 – The function $f(x)$ is exponential (negative exponential distribution)

Function $f(x)$	$f(x)=ka^{-hx}$	$\omega=\infty$
Density $f(x)$	$f(x)=ke^{-2kx}$	$a=e^{2k/h}$
Exercise probability $F(x=M)$	$F(M)=\frac{e^{-2kM}}{2}$	
Pricing kernel	$\frac{\partial P_1}{\partial M}=-F(M)=-\frac{e^{-2kM}}{2}$	$P_1(M=\omega)=0\Rightarrow c=0$
Call	$P_1=\frac{e^{-2kM}}{4k}$	
ATM Call/Put	$P=\frac{1}{4k}$	
Relation between ATM Call and general Call	$P_1=e^{-\frac{1}{2}\frac{M}{P}}P$	

5.5 The normal law of error

The most exciting specification of $f(x)$ is the law of error (*Fehlergesetz*) defined by $f(x)=\frac{h}{\sqrt{\pi}}e^{-h^2x^2}$ [53]. Unlike the previous specifications of $f(x)$, this is now a direct specification of the probability density. Reasoning that market variations above and below the forward price B can be regarded as deviations from the markets' most favorable outcome, Bronzin suggest to use the law of error as a very reliable law to represent error probabilities[54]. Of course, the density corresponds to a normal distribution with zero mean and a standard deviation of $\sigma_{err}=\frac{1}{h\sqrt{2}}$. Or alternatively, setting $h=\frac{1}{\sigma\sqrt{2}}$ gives us the normal $N\{0,\sigma^2\}$.[55]

In order to compare the ATM option price with the previous Section, it is necessary to have equal variances. The variance of the exponential distribution is given by

$$Var_{\exp}(x)=\int_0^\infty x^2 f(x)dx=\int_0^\infty x^2ke^{-2kx}dx$$

Applying the formula $\int_0^\infty x^n e^{-ax}dx=n!a^{-n-1}$ gives

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[53] The (normal) law of error should not be confused with error *function* which is an integral defined by $erf(x)=2\int_0^x \frac{1}{\sqrt{\pi}}e^{-t^2}dt$, related to the cumulative standard normal $N\{\}$ by $erf(x)=2\left[N\{x\sqrt{2}\}-0.5\right]$.

[54] Original text: „*Indem wir uns also die Marktschwankungen über oder unter* B *gleichsam als Abweichungen von einem vorteilhaftesten Werte vorstellen, werden wir versuchen, denselben die Befolgung des Fehlergesetzes ... vorzuschreiben, welches sich zur Darstellung der Fehlerwahrscheinlichkeiten sehr gut bewährt hat; ...*" (p. 74).

[55] As a historical remark, the analytical characterization as well as the terminology related to the "normal" distribution was very mixed until the end of the 19th century; while statisticians like Galton, Lexis, Venn, Edgeworth, and Pearson have occasionally used the expression in the late 19th century, it was adopted by the probabilistic community not earlier than in the 1920s. Stigler (1999), pp. 404-415, provides a detailed analysis of this subject.

$$Var_{\exp}(x) = k\int_0^\infty x^2 e^{-2kx}dx = k \times \left\{2 \times (2k)^{-2-1}\right\} = \frac{2k}{(2k)^3} = \frac{1}{(2k)^2}$$

so that the volatility is

(8)… $$\sigma_{\exp}(x) = \frac{1}{2k}$$

The variance of the error distribution can be computed by the same procedure; alternatively one can easily substitute the parameter $h = \frac{1}{\sigma\sqrt{2}}$ in the function to get

$$f(x) = \frac{h}{\sqrt{\pi}}e^{-h^2x^2} = \frac{\frac{1}{\sigma\sqrt{2}}}{\sqrt{\pi}}e^{-\frac{1}{2\sigma^2}x^2} = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2}$$

which is the density function of a normally distributed variable with zero mean and standard deviation σ. Solving $h = \frac{1}{\sigma\sqrt{2}}$ for σ gives

(9)… $$\sigma_{err}(x) = \frac{1}{h\sqrt{2}}$$

which shows the standard deviation of the error distribution implied by a specific choice of parameter h. Since h is inversely related to the standard deviation of the distribution, it measures the precision of the observations, and is called *precision modulus;* see Johnson/ Kotz/ Balakrishnan (1994), p. 81.

The relationship between the volatility of the exponential and the error distribution is then given by the equality $2k = h\sqrt{2}$ or

(10)… $$h = k\sqrt{2}\,.$$

The implied ATM option price is therefore

$$P_{err}\left(h = k\sqrt{2}\right) = \frac{1}{2k\sqrt{2}\sqrt{\pi}} = \frac{1}{k\sqrt{8\pi}} = \frac{1}{5.013 \times k}$$

which is only about 80% of the exponential ATM option price $P_{\exp} = \frac{1}{4k}$. This is not surprising: compared to the exponential distribution, the error (or normal) distribution has more weight around the mean and less around the tails – given the same standard deviation.

It is also interesting to compare the ATM option price with the quadratic case examined two Sections before. For this purpose we need to know the relationship between the parameters h and ω; combining $h = k\sqrt{2}$ with $k = \frac{3}{2\omega}$ which was used as condition of consistency between the quadratic and exponential function (in the previous section), this gives $h = \frac{3}{2\omega}\sqrt{2} = \frac{\sqrt{4.5}}{\omega}$. Inserting this in $P = \frac{1}{2h\sqrt{\pi}}$ yields

$$P_{err} = \frac{1}{2\frac{\sqrt{4.5}}{\omega}\sqrt{\pi}} = \frac{\omega}{7.51988482}$$

which is only approx. 6% more than the price of the respective option priced with the quadratic function, $P_q = \frac{\omega}{8}$. The similarity of the option prices is not surprising given the similarity of the two densities; see **Figures 2b/2d.**

The impact of the moneyness is less obvious than in the former cases. This will be discussed below when we compare the formula with the Black-Scholes case.

Table 5 – The function $f(x)$ is the normal law of error

Density $f(x)$	$f(x)=\frac{h}{\sqrt{\pi}}e^{-h^2x^2}$	$\omega=\infty$
Exercise probability $F(x=M)$	$F(M)=\psi(hM)$ $\psi(\varepsilon)=\frac{1}{\sqrt{\pi}}\int_{\varepsilon}^{\infty}e^{-t^2}dt$	
Pricing kernel	$\frac{\partial P_1}{\partial M}=-F(M)=\ldots$	$P_1(M=\omega)=0\Rightarrow c=0$
Call	$P_1=\frac{e^{-M^2h^2}}{2h\sqrt{\pi}}-M\,\psi(hM)$	
ATM Call/Put	$P=\frac{1}{2h\sqrt{\pi}}$	

5.6 A comparison with the Black-Scholes model

a) Normal versus lognormal market price

Obviously, the specification of the pricing function in the previous section is particularly interesting, because it promises a direct link to the celebrated Black-Scholes model[56]. As seen before, setting $h=\frac{1}{\sigma\sqrt{2}}$ in the error function generates the normal distribution. The problem is, however, that the Black-Scholes model assumes a normal distribution for the *log*-prices, while Bronzin makes this assumption for the price *level* itself. In terms of the underlying stochastic processes, Bronzin's distribution can be regarded[57] as the result of an arithmetic Wiener process, while the Black-Scholes model relies on a geometric Wiener process.

[56] We adopt the common terminology in using „Black-Scholes“ for the models developed by Black/ Scholes (1973) and Merton (1973).
[57] There is however no reference to a specific stochastic process in Bronzin's text.

Since there is an immediate link between the two processes, why not interpreting Bronzin's price levels as log-prices? This is, however, not adequate in the option pricing framework because the value of options is a function of the payoff emerging from the (positive) difference between settlement *price* and exercise *price* of the option, not their logarithms. In this respect, the approach of Bronzin is the same as the one of Bachelier. It was only Samuelson (1973) who corrected the possibility of negative prices in the Bachelier model by modeling the Wiener process of speculative price in logs instead of levels[58].

More precisely, the analytical complication comes from the following point. The pricing function for a call option with exercise price $B+M$ in the Bronzin setting is

$$(11) \quad P_1 = \int_M^{\infty} (\tilde{x} - M) f(x) dx, \quad f(x) = \frac{h}{\sqrt{\pi}} e^{-h^2 x^2} = \frac{1}{\sigma(x)\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x}{\sigma(x)}\right)^2}$$

where $\tilde{x}$ is the deviation of the market price at maturity from the forward price, described by the error distribution, or the normal, with zero mean and standard deviation $\sigma(x) = \frac{1}{h\sqrt{2}}$. In contrast, the Black-Scholes solution assumes a lognormal distribution for $\tilde{x}$. How does this change the shape of the option formula? This is addressed in the next section.

b) Deriving the Black-Scholes formula from the Bronzin equation

We show how to rewrite the Bronzin's equation to get the Black-Scholes formula. For this purpose, we replace

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[58] To clarify the terminology: either the log (more precisely: the natural logarithm) of the stock price follows an arithmetic Wiener process and is normally distributed, or the stock price itself follows a geometric Wiener process and is *log*normally distributed.

(12)… $\tilde{x} - M = [\tilde{S}_T - B] - [K - B] = \tilde{S}_T - K$,

and assume that $\tilde{S}_T$ is *log*normally distributed, which we write in terms of the standard normal $\tilde{z}$ as

(13a)… $\tilde{S}_T = S_t e^{\mu(T-t)+\sigma\tilde{z}\sqrt{T-t}}$, with $\mu = \dfrac{E\left[\ln\left(\dfrac{S_T}{S_t}\right)\right]}{T-t}, \sigma^2 = \dfrac{Var\left[\ln\left(\dfrac{S_T}{S_t}\right)\right]}{T-t}$.

Adapting the risk-neutral valuation approach of Cox/ Ross (1976), the drift of the log stock price changes can be replaced by $\mu = r - \frac{1}{2}\sigma^2$. In order to facilitate the comparison with Bronzin, we subsequently assume an interest rate of zero and one time unit to maturity, $T - t = 1$ (e.g. one year if volatility is measured in annual terms). The forward price is then equal to the current stock price, implying

(13b)… $\tilde{S}_T = Be^{-\frac{1}{2}\sigma^2+\sigma\tilde{z}}$.

The Black-Scholes valuation equation can then be written as

(14)… $P_1 = \int_{-z_2}^{\infty}\left(\underbrace{Be^{-\frac{1}{2}\sigma^2+\sigma\tilde{z}}}_{S_T} - K\right)N'(z)dz$,

where the remaining task is to adjust the lower integration boundary, here denoted by $-z_2$ in anticipation of the Black-Scholes model. For this task, we just have to transform the probability range of the normal $\tilde{x}$, $pr(\tilde{x} > M)$, to a new range $pr(\tilde{S}_T > K)$ expressed relative to the standard normal density $N'(z)$. Notice that $pr(\tilde{S}_T > K)$ is equal to $pr(\ln S_T > \ln K)$, and that $\ln(S_T)$ is normally distributed with mean $\ln S_0 - \frac{1}{2}\sigma^2 = \ln B - \frac{1}{2}\sigma^2$ and standard deviation σ. Thus we can standardize both sides of the inequality $pr(\ln S_T > \ln K)$ to get

$$pr\left(\underbrace{\frac{\ln \tilde{S}_T - \left[\ln B - \tfrac{1}{2}\sigma^2\right]}{\sigma}}_{\tilde{z}} > \frac{\ln K - \left[\ln B - \tfrac{1}{2}\sigma^2\right]}{\sigma}\right)$$

where $\tilde{z}$ is the standard normal. The expression on the r.h.s. can be written as

(15)… $$\frac{\ln K - \left[\ln B - \tfrac{1}{2}\sigma^2\right]}{\sigma} = -\frac{\ln B - \ln K - \tfrac{1}{2}\sigma^2}{\sigma} = -\frac{\ln \frac{B}{K} - \tfrac{1}{2}\sigma^2}{\sigma} \equiv -z_2$$

which is exactly the Black-Scholes boundary typically expressed as

$$pr(\tilde{z} > -z_2) = pr(\tilde{z} < z_2) \equiv N(z_2).$$

Summing up, we have shown that the Bronzin equation (11) can be easily transformed to the Black-Scholes model if the stock price $B + \tilde{x} = \tilde{S}_T$ is specified as a lognormal instead of a normal variable and the integration boundary is adjusted correspondingly. Thus, the pricing relationship looking most similarly to the Bronzin equation is

$$P_1 = \int_K^\infty \left(\tilde{S}_T - K\right) L'(S_T)\, dS_T = \int_{-z_2}^\infty \left(Be^{-\tfrac{1}{2}\sigma^2 + \sigma\tilde{z}} - K\right) N'(z)\,dz, \quad z_2 = \frac{\ln \frac{B}{K} - \tfrac{1}{2}\sigma^2}{\sigma}$$

The explicit solution using the standard Black-Scholes procedure in transforming the integral[59] is

$$P_1 = B\,N\left\{z_2 + \sigma\sqrt{T-t}\right\} - K\;N\left\{z_2\right\}$$

or re-adapting time and interest,

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[59] see e.g. James (2003), pp. 299-309.

$$P_1 = Be^{-r(T-t)}\, N\left\{z_2 + \sigma\sqrt{T-t}\right\} - Ke^{-r(T-t)}\, N\{z_2\}$$

where $Be^{-r(T-t)} = S_t$ can also be written as the current stock price. This derivation shows that Bronzin's valution equation (11) is fully consistent with the Black-Scholes, and, respectively, the Black (1976) forward price based valuation models. It is also a risk-neutral valuation approach – he makes no assumptions on preferences or expected values – simply because the option price relies on the forward price and zero (random) deviations from there.

c) The "Bronzin" style Black-Scholes formula

It is furthermore interesting to examine how Bronzin solves his valuation integral (11), and to adapt the procedure to the Black-Scholes case. The integral is split up in two parts

$$P_1 = \int_M^\infty (\tilde{x} - M) f(x) dx = \int_M^\infty \tilde{x}\; f(x) dx - M \int_M^\infty f(x) dx, \quad \text{with } f(x) = \frac{h}{\sqrt{\pi}} e^{-h^2 x^2}$$

The first integral is the conditionally expected market price at maturity (corrected by the forward price) – conditional upon option exercise. The second integral is the exercise probability. No explicit solution is available for the second integral, but Bronzin provides a table for alternative values for $\psi(\varepsilon) = \frac{1}{\sqrt{\pi}} \int_\varepsilon^\infty e^{-t^2} dt$ in an Appendix (pp. 84-85).

Notice that t exhibits a standard deviation of $\frac{1}{\sqrt{2}}$, and it is related to the standard normal by

$$\text{(16a)} \ldots \frac{h}{\sqrt{\pi}} \int_M^\infty e^{-h^2 x^2} dx = \underbrace{\frac{1}{\sqrt{\pi}} \int_{hM}^\infty e^{-t^2} dt}_{\equiv \psi(hM)} = \frac{1}{\sqrt{2\pi}} \int_{hM\sqrt{2}}^\infty e^{-\frac{1}{2}z^2} dz = \frac{1}{\sqrt{2\pi}} \int_{M/\sigma(x)}^\infty e^{-\frac{1}{2}z^2} dz = N\left(-\frac{M}{\sigma(x)}\right).$$

This relationship will be useful below. In contrast to the second integral, the first integral $\int_M^\infty \tilde{x}\ f(x)dx = \frac{h}{\sqrt{\pi}}\int_M^\infty \tilde{x}\ e^{-h^2x^2}dx$ has an explicit solution[60], which is recognized by Bronzin (p. 76), namely,

$$(16b)\ldots\ \frac{h}{\sqrt{\pi}}\int_M^\infty \tilde{x}\ e^{-h^2x^2}dx = \frac{h}{\sqrt{\pi}}\left\{\frac{1}{2h^2}e^{-M^2h^2}\right\} = \frac{1}{2h\sqrt{\pi}}e^{-M^2h^2}\ .$$

The option price is therfore

$$(17)\ldots\ P_1 = \frac{1}{2h\sqrt{\pi}}e^{-M^2h^2} - M\,\psi(hM), \qquad \psi(\varepsilon) = \frac{1}{\sqrt{\pi}}\int_\varepsilon^\infty e^{-t^2}dt\ .$$

Again, this formula enables to separate between the impact of volatility ($M = 0$) and intrinsic value on option price, although the distinction is not so clear as in the previous formulae. Notice that the first term adds the same positive amount to the option value irrespective whether the option is in- or out-of-the money ($M \neq 0$).

Based on this solution, we could also try to write the Black-Scholes formula in the "Bronzin style". We rewrite (14) as

$$P_1 = \int_{-z_2}^\infty \left(B\left(e^{-\frac{1}{2}\sigma^2+\sigma\tilde{z}} - 1\right) - \underbrace{(K-B)}_{M} \right) N'(z)dz$$

where the exponential expression is approximated by

$$e^{-\frac{1}{2}\sigma^2+\sigma\tilde{z}} \approx 1 + \left(-\tfrac{1}{2}\sigma^2 + \sigma\tilde{z}\right) + \tfrac{1}{2}\left(-\tfrac{1}{2}\sigma^2 + \sigma\tilde{z}\right)^2 + -\ldots = 1 + \sigma\tilde{z} + -\ldots$$

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[60] The solution of the integral $\int x\,e^{ax^2}dx$ is $\frac{1}{2a}e^{ax^2}$. Setting $a = -h^2$ and evaluating the integral at the boundaries $[M,\infty]$, we find $\int_M^\infty x\,e^{-h^2x^2}dx = \left[\frac{1}{-2h^2}e^{-h^2x^2}\right]_M^\infty = -\frac{1}{-2h^2}e^{-h^2M^2}$.

where we neglect asymptotically vanishing terms. We then get

$$P_1 = B\sigma \int_{-z_2}^{\infty} \tilde{z}\left(\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\right) dz - M \; N\{z_2\}$$

or written in a slightly more complicated way

$$P_1 = B\sigma \int_{-z_2}^{\infty} \tilde{z}\left(\frac{\frac{1}{\sqrt{2}}}{\sqrt{\pi}} e^{-\left(\frac{1}{\sqrt{2}}\right)^2 z^2}\right) dz - M \; N\{z_2\}$$

which is the same as setting $h = \frac{1}{\sqrt{2}}$ in the Bronzin solution (16b). The option price is thus

$$P_1 = B\sigma \frac{e^{-(-z_2)^2\left(\frac{1}{\sqrt{2}}\right)^2}}{2\left(\frac{1}{\sqrt{2}}\right)\sqrt{\pi}} - M \; N\{z_2\} = B\sigma \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z_2)^2} - M \; N\{z_2\}$$

which can also be written as

$$(18a)\ldots \quad P_1 = B\sigma \; N'\{z_2\} - M \; N\{z_2\}, \qquad z_2 = \frac{\ln\frac{B}{K} - \frac{1}{2}\sigma^2}{\sigma}$$

This is the Bronzin-style "Black-Scholes" model. The value of the put option is then simply

$$(18b)\ldots \quad P_2 = P_1 + M = B\sigma \; N'\{z_2\} - M \; N\{z_2\} - (-M) = B\sigma \; N'\{z_2\} - M \; [N\{z_2\} - 1]$$

Notice that these expressions are approximations – but they highlight some interesting aspects of the Black-Scholes formula. The exact relation to the Bronzin-model (17) is straightforward. First, approximate

$$z_2 = \frac{\ln\frac{B}{K} - \frac{1}{2}\sigma^2}{\sigma} = \frac{-\ln\frac{M+B}{B} - \frac{1}{2}\sigma^2}{\sigma} = \frac{-\ln\left(1+\frac{M}{B}\right) - \frac{1}{2}\sigma^2}{\sigma} \approx \frac{-M/B}{\sigma} = -\frac{M}{B\sigma}$$

and replace $B\sigma = \sigma(x)$. It was shown in equation (16a) that $N(z_2) = N\left(-\frac{M}{\sigma(x)}\right) = \psi(hM)$ which shows the equivalence of the second term in the pricing equation. The equivalence of the first term requires exactly the same substitutions and approximations, i.e.

$$B\sigma\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(z_2)^2} = \frac{\sigma(x)}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(-\frac{M}{\sigma(x)}\right)^2} = \frac{1}{2h\sqrt{\pi}}e^{-h^2M^2}$$

just by recognizing $h = \frac{1}{\sigma(x)\sqrt{2}}$. This completes the formal equivalence between the Bronzin and Black-Scholes model: The two models just differ with respect to the distributional assumption of the underlying market price: Bronzin assumes a normal distribution for the price level (respectively, its deviation from the forward price), while Black-Scholes assume a normal distribution for the *log* price (in addition, with time-proportional moments). But the rest of the two models is *identical,* including the risk-neutral valuation approach (preference-free mean of the pricing density) – which is an amazing observation.

d) A simple expression (approximation) for at-the-money options

Equation (18a) can also be used to get a "back on the envelope" formula for ATM Black-Scholes prices. We set $M = 0$ and $z_2 = -\frac{1}{2}\sigma$ to get $P_1 = B\sigma\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{8}\sigma^2}$. For conventional volatilities, the exponent is extremely small, so that the exponential expression is close to unity (e.g. if the volatility is 20%,the expression is 0.995). So we get

(19) $P_1 \approx \frac{B\sigma}{\sqrt{2\pi}} = 0.399 \times B\sigma$

which corresponds to Bronzin's ATM option value; substituting $h = \frac{1}{\sigma(x)\sqrt{2}}$ in his equation (44) gives

$$P_1 = \frac{1}{2h\sqrt{\pi}} = \frac{1}{2\left(\frac{1}{\sigma(x)\sqrt{2}}\right)\sqrt{\pi}} = \frac{\sigma(x)}{\sqrt{2\pi}}$$

Notice, however, that Bronzin's expression is exact, while ours (equation 19) is an approximation. The same expression can be found in Bachelier (1900), after appropriate adjustments[61].

Thus, the (relative) price of an ATM option is 39.9% or 40% of the absolute price volatility. If the forward rate has a volatility of 20%, then the value of an ATM call or put option with 1 year to maturity is approx. 8% of the forward price, the price of a respective 3m option is 4%.

5.7 The binomial distribution ("Bernoulli theorem")

While sections 2 through 6 in the 2nd chapter of part II in Bronzin's book are direct specifications of the pricing density $f(x)$, the approach taken in his final section 7 is slightly different. It can be understood as a mere specification of the (inverse) volatility factor h in

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[61] See his 2nd equation on p. 51, $a = k\sqrt{t}$, where a is the price of an ATM option (in French: *prime simple*) and t is the time to maturity. Denoting the standard deviation of the normally distributed stock price changes over the time period t by $\sigma(x)\sqrt{t}$, it follows immediately that k must be specified by $k = \frac{\sigma(x)}{\sqrt{2\pi}}$ in his probability density function (e.g. see his 5th equation on p. 38). It then follows that $a = \frac{\sigma(x)\sqrt{t}}{\sqrt{2\pi}}$, which is our expression, except that the volatility has an explicit time dimension in Bachelier's distribution.

the error function. The starting point of his analysis is almost identical to the binomial model of Cox/ Ross/ Rubinstein (1979). Assuming that s (consecutive) price movements[62] are governed by "two opposite events" (e.g. market ups and downs) with probability p and q, which can be thought as Bernoulli trials. The expected value of the distribution is sp (or alternatively, sq)[63]. Of course, the events can be scaled arbitrarily by choosing the parameter s appropriately. Therefore, one of the expected values (which one is arbitrary) can be set equal to the forward price, e.g. $B = sp$. The price distribution can then be understood as being generated by cumulative deviations of market events from their most likely outcome, the forward price. The standard deviation of this distribution is $\sqrt{spq} = \sqrt{Bq}$.

If $\tilde{x}$ denotes the price deviations between the market price and the forward price, Bronzin uses the following expression to describe the probability that $\tilde{x}$ is in the interval $[0; x^*]$ [64]

(20)… $$\frac{1}{\sqrt{2\pi}}\int_0^{z^*} e^{-\frac{1}{2}z^2}dz + \frac{e^{-z^2}}{\sqrt{2\pi}\sqrt{Bq}}, \text{ with } z^* = \frac{x^*}{\sqrt{Bq}},\ \tilde{z} = \frac{\tilde{x}}{\sqrt{Bq}}$$

and neglects the second expression in his subsequent analysis (the term being "of secondary importance", which is of course not exactly true). He then notices that for $h = \frac{1}{\sqrt{2qB}}$, or in our own notation, for

(21) $$\sigma(x) = \frac{1}{h\sqrt{2}} = \sqrt{qB}$$

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[62] Again, there is no reference to a time dimension in Bronzin's approach. In the Cox/ Ross/ Rubinstein (1979) setting, these would be interpreted as consecutive market movements. In the Bronzin setting, the binomial approach is just used to characterize the deviations from the expected (i.e. forward) price.

[63] Original text: „*... so stellen* ps *resp.* qs *die wahrscheinlichsten Wiederholungszahlen der betrachteten Ereignisse dar*" (p. 80).

[64] We use a simpler notation than Bronzin, who operates with the error function; see his equations (47) and (50).

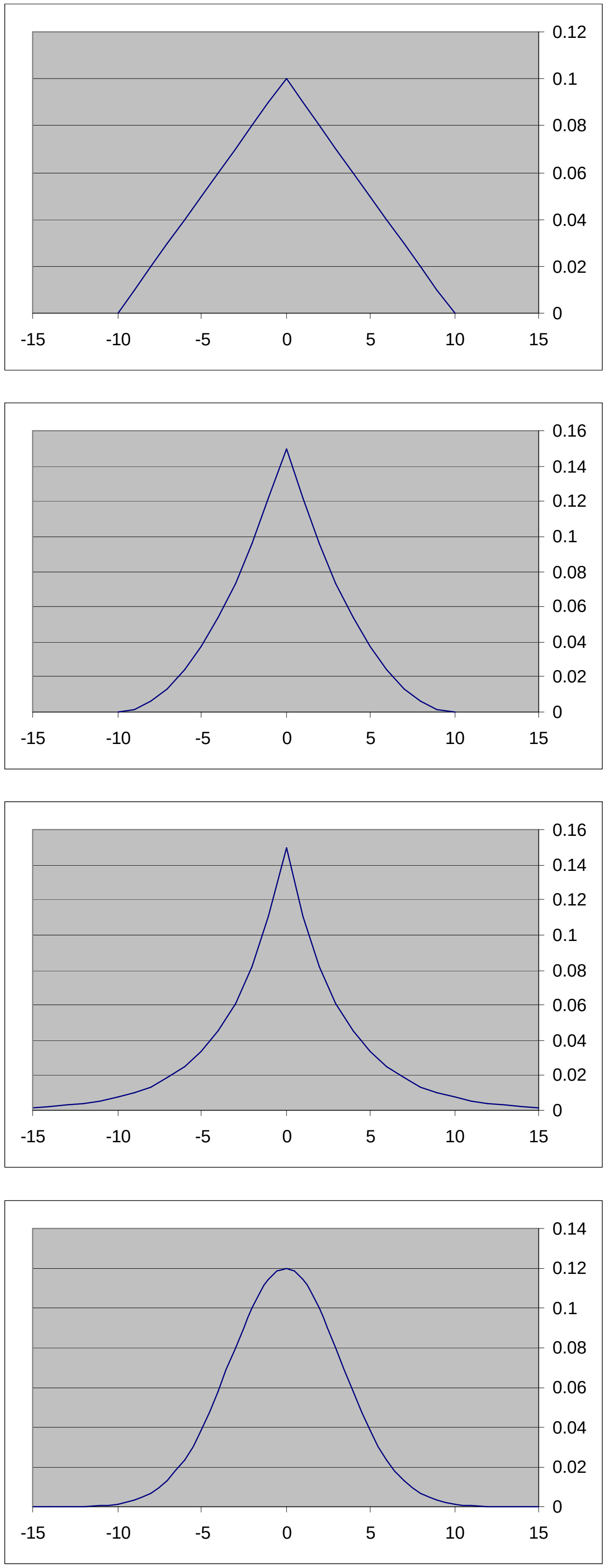
this is the same integral as in the previous section where $f(x)$ was specified by the normal density. He concludes that the application of the Bernoulli theorem to market movements leads to the same results as the application of the error law[65]. Given the asymptotic properties of the binomial distribution, this is of course not a surprising result. It is, however, interesting to notice that he treats the Bernoulli model as a way to motivate the "limiting" case of the error function in the same way as Cox/ Ross/ Rubinstein (1979) demonstrate that their binomial model converges to the Black/Scholes model in the limiting case. Finally it is interesting to notice that Bachelier (on pp. 38 ff) also uses a binomial binomial to retrieve the properties of the Wiener process developed before.

Bronzin also recognizes that the volatility, respectively his h, is not a variable which can be directly observed. He repeatedly stresses this point by arguing that this parameter needs to be empirically estimated for each underlying - again on p. 81. However, he recognizes that by specifying the expected value of his binomial distribution by $sp = B$, then the only part which remains unspecified in his volatility expression is the q parameter; see equation (21). If the "preference based" q parameter would be known, then the volatility could be directly inferred from the forward price B. E.g. if $q = \frac{1}{2}$, then the volatility would be the square root of half the forward price, $\sigma(x) = \sqrt{\frac{B}{2}}$: see Bronzin's equation (51a). He is surprised, or puzzled, about this finding (p. 82) and notices that the volatility of market prices is likely to depend on many other factors than the observed forward price. However, it may be useful to read the result of equation (21) in a different way, namely by understanding q as the endogenous variable. It then implies that $q = \frac{\sigma^2(x)}{B}$, i.e. increasing the variance of the under-

[65] Orig. text: "*... so ersehen wir aus der vollkommenen hier herrschenden Analogie, dass uns die Anwendung des Bernoullischen Theorems auf die Marktschwankungen zu demselben Resultate, wie die Annahme der Befolgung des Fehlergesetzes, führt*" (p. 81).

lying while leaving B increases probability q. This is by no means a surprising result. We just have to re-interpet Bronzin's probabilities as risk-neutral probabilities, which is legitimate as discussed earlier (Section 4.3). Increasing the variance while leaving the stock price and interest rate (and thus, the forward price) constant, implies a shift of the risk-neutral density to the left (the risk-neutral mean of the distribution falls), which means a higher probability for bad states. This is exactly what a higher probability q means; remember that the forward price was matched with the expected value of the distribution sp, so that p are the probabilities of the "good" states (market event) by definition.

Figures 2a-2d Specifications of the pricing density function (linear, quadratic, exponential, error function and the associated densities)



6. *Valuation of Repeat-Options ("Noch"-Geschäfte)*

This section reviews the valuation of a specific type of option contract which was apparently important in the days of Bronzin. In brief terms, the holder of a forward contract acquires an option, by paying a premium N_m (the *Noch*-premium), to repeat the transaction m times at maturity. In case of a long forward contract, the holder acquires the right to increase the original contract size by the multiple m of the original contract size, i.e. to buy additional shares at maturity of the forward contract. The exercise price is set above the forward price, namely at $B + N_m$. Equivalently, the holder of a short forward contract acquires an option to sell an additional quantity of m times the original contract size at maturity; the exercise price is fixed below the forward price, at $B - N_m$. We will call the first option contract a repeat-call option, the second contract a repeat-put option.

Unlike in a standard option contract, the premium N_m serves a double function: It is the option price paid in advance, but also stands for the premium added to (or subtracted from) the forward price in fixing the exercise price of the option. This double function complicates the determination of the fair premium[66]. A fundamental restriction in computing the premium is $N_m = mP_1$, where P_1 is the price of a simple "skewed" (non-ATM) call option. Bronzin shows that this condition must hold by arbitrage (pp. 48-50, equation 15). More specifically, the valuation problem for a repeat-call option can be stated as[67]

(22)… $$N_m = mP_1 = m\int_{N_m}^{\infty}\left(\tilde{x} - N_m\right)f\left(x\right)dx,$$

[66] Obviously, it is fairly arbitrary that the premium of the option has to be identical to the "markup" to be paid at exercise. But it seems that this was a business standard.

[67] In the following, we adapt the notation of Bronzin, except that we add the subscript m to the repeat-option premium N.

where $f(x)$ is the pricing density, as discussed in Section 5. The following remark on $(\tilde{x} - N_m)$ could be useful: Remember that $\tilde{x}$ denotes the deviation of the market price at maturity from the forward price; according to our contractual characterization of the repeat-option, the exercise price consists of the forward price plus (minus) the premium, $K = B \pm N_m$. So, the skewness of the contract, characterized by M, is entirely determined by the premium. Hence, the payoff of the contract is given by

$$x - M = [S_T - B] - [K - B] = [S_T - B] - [(B + N_m) - B] = x - N_m$$

which is the expression in our equation (22).

Repeat-options are analyzed throughout his book. A description of the contracts and some fundamental hedging relationships can be found on pp. 30-37; general pricing relationship are derived on pp. 48-50; and concrete pricing solutions for the various specifications of $f(x)$ are provided throughout his second chapter of part II.

Pure inspection of our equation (22) suggests that finding explicit solutions for the premium N_m is not an easy task: It shows up on the l.h.s. of the equation, and twice on the r.h.s. – within the payoff function and on the integration boundary. For very simple specifications of the pricing density, explicit solutions can be easily derived, but approximations or numerical solutions are inevitable for even slightly more complicated choices. An extremely elegant solution is provided by Bachelier (1900) for the case of normal distributions; we will discuss this shortly.

For illustrative purposes, we only briefly outline the solution for the simplest case, when $f(x)$ is assumed to be constant within the interval $[-\omega;+\omega]$. According to **Table 1**, the option price for the constant case is $P_1 = \frac{(\omega - M)^2}{4\omega}$. In order to get the repeat-option premium N_m, the skewness of the option must be adjusted to $M = N_m$, and by equation (23) the expression must be multiplied by m:

$$N_m = mP_1 = m\frac{(\omega - N_m)^2}{4\omega}$$

This is a quadratic equation in our unknown N_m, which can be easily solved; however, ω would remain unspecified in this setting. It will be useful to substitute this parameter by the (possibly observable) ATM option price given by $P = \frac{\omega}{4}$, which results in

(23)… $$\frac{N_m}{P} = m\left(1 - \frac{1}{4}\frac{N_m}{P}\right)^2$$

It turns out that the structure of this expression (relating the premium to the ATM option price) is very useful throughout the analysis, particularly for computational purposes. In our simple setting here, the solution is given by

$$\frac{N_m}{P} = \frac{4\left(m + 2 - 2\sqrt{1+m}\right)}{m}$$

which is Bronzin's equation (7a) on p. 59. Alternative integer values for m can now be plugged in this expression to get the fair premium for 1-time, 2-times, 3-times etc. repeat-options, e.g.

$$N_1 = \frac{4\times\left(1 + 2 - 2\times\sqrt{1+1}\right)}{1} = 4\times\left(3 - 2\times\sqrt{2}\right) = 0.6863\,P$$

$$N_2 = \frac{4\times\left(2 + 2 - 2\times\sqrt{1+2}\right)}{2} = 2\times\left(4 - 2\times\sqrt{3}\right) = 1.072\,P$$

and so on. It is, of course, interesting to notice that the premium does not increase proportionally with the number of repeats. Specifically, the relation between N_2 and N_1 is

$$N_2 = 1.562\, N_1$$

which is a figure that attracts a lot of attention in Bronzin's analysis. Alternatively, one could also be interested in finding the number of repeats which are necessary[68] to equate the premium to the price of an ATM option, i.e. $\frac{N_m}{P} = 1$; we just have to insert this ratio in equation (24) and solve for m:

$$m = \frac{\frac{N_m}{P}}{\left(1 - \frac{1}{4}\frac{N_m}{P}\right)^2} = \frac{1}{\left(1 - \frac{1}{4}\right)^2} = \frac{1}{9/16} = 1.777$$

An overview on the solutions for the other specifications of the pricing density can be found in **Table 6**. The amazing observation is how similar the numerical values are (see the bold figures) given the different shape of the distributions. Bronzin shows repeatedly puzzled about this "remarkable", "strange" coincidence.

It is interesting to notice that Bachelier analyzes the same contracts on pp. 55-57, called *options d'ordre n* (in contrast to *primes* analyzed otherwise)[69]. He provides a particularly elegant solution to the pricing problem. Throughout his analysis he assumes that the (absolute) stock price changes are characterized by a normal (with mean zero and annualized volatility[70] $k\sqrt{2\pi}$). He then uses an extremely useful approximation of the normal integral which results in

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[68] This is somehow unrealistically from a practical point of view, because the solution will not be an integer in general.
[69] There is a minor institutional difference between the contract specification, because the "fixed part" (i.e. the forward contract) seems to exhibit the same exercise price as the *m* options, which is somehow strange.
[70] Notice that this is not "our" k from the exponential function.

$$N_m = P\left(\frac{m+2}{m}\pi - \sqrt{\left(\frac{m+2}{m}\pi\right)^2 - 4\pi}\right)$$

(see his 5th equation of p. 56); we have changed the symbols to match our notation. Plugging in the desired parameters m, gives the following values:

m	*1*	*2*	*3*	*4*	*5*	*10*
Bachelier	0.6921	1.0955	1.3825	1.6075	1.7948	2.4870
Bronzin	0.6919	1.0938				

which shows that the values for $m = 1, 2$ are virtually identical. Obviously, the Bachelier solution is much more elegant and allows to directly compute the premium for an arbitrary number of multiples. It is obvious that the increase of the premium is degressive with respect to m.

Table 6 Valuation characteristics of repeat-options (*Noch-Geschäfte*)

	constant	linear	quadratic	exponential	error function
Reference	pp. 59-61	pp. 63-65	pp. 68-69	pp. 71-74	pp. 76-80
$\frac{N_m}{P}$	$m\left(1-\frac{N_m}{4P}\right)^2$	$m\left(1-\frac{N_m}{6P}\right)^3$	$m\left(1-\frac{N_m}{8P}\right)^4$	$me^{-\frac{1}{2}\frac{N_m}{P}}$	$m\left(e^{-\frac{1}{2}\left(\frac{N_m}{\sqrt{2\pi}P}\right)^2} - \frac{N_m}{P}\varphi\left\{\frac{N_m}{2P\sqrt{\pi}}\right\}\right)$
N_1	0.6864 P	0.6928 P	0.6952 P	0.70355 P	0.6919 P
N_2	1.672 P	1.0936 P	1.104 P	1.1345 P	1.0938 P
$\frac{N_2}{N_1}$	**1.562**	**1.578**	**1.588**	**1.612**	**1.581**
$m\left(\frac{N_m}{P}=1\right)$	**1.777**	**1.728**	**1.7059**	**1.6487**	**1.7435**

All figures are adapted from Bronzin, no own computations.

7. *Option pricing in historical perspective, and an evaluation of Bronzin's contribution*

Judgements about scientific originality are always difficult with a delay of a century, in a field which has progressed so rapidly as option pricing, and where statistical and stochastic methods are used which were hardly developed at this time. It is even questionable whether scientific originality is a fair criterion to apply – because nothing is known about its purpose or target audience. Given that he published it as a "professor", and given that he has published a textbook on actuarial theory for beginners two years before (Bronzin 1906), it may well be that he regarded his option theory as a textbook, or a mixture between textbook and scientific monograph. Finally, Bronzin did definitively not overstate his own contribution – he even understates it by regularly talking about his "booklet" (*Werkchen*) when referring to it[71]. Why he was talking about his textbook as a "booklet" is an open question: Was it, because it was not good for his reputation as an academic to write about financial mathematics – or worse, on a topic typically associated with speculation? Was it because the subject was too far away from his profession as a professor for actuarial theory? We do not know. Further research has to be done.

The originality in the field of option pricing is difficult to assess anyway. Who deserves proper credit for the Black-Scholes model? The early Samuelson (1965) paper contains the essential equation[72]. Even more puzzling is a footnote in the Black-Scholes paper (p. 461) where the authors acknowledge a comment by Robert Merton suggesting that if the option hedge is maintained continuously over time, the return on the hedged position becomes certain. But it is the notion of the riskless hedge which makes the essential difference between

[71] The German word is actually a funny combination of *Work* which means, in an academic setting, a substantial contribution, while the ending *...chen* is a strong diminutive.
[72] Or to use Samuelson's own wording: "Yes, I had the equation, but "they" got the formula…"; see Geman (2002).

Black/ Scholes and the earlier Samuelson and Merton/ Samuelson models[73], [74]! Surprisingly enough that Merton was kind enough to delay publication of his (accepted) 1973 paper until Black/ Scholes got theirs accepted[75].

An open question is to what other publications Bronzin is referring to: He surely knew the most important publications in German language about probability and options. Options ("Prämiengeschäfte") were common instruments at this time at the stock exchanges in the German spoken part of Europe. Their were many different forms. And at least about the legal aspect of the options different books containing financial transactions have been published. But the mathematical background of options didn't seemto be an issue. Another question is, whether Bronzin knew about Bachelier's work. *Honni soit qui mal y pense …* - but extensive quoting was not the game at the time anyway. Bachelier did not quote any of the earlier (but admittedly, non mathematical) books on option valuation either. For example, the book of Regnault (1863) was widely used and contains the notion of random walk, the Gaussian distribution, the role of volatility in pricing options, including the square-root formula[76]. According

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[73] To be precise, the notion of a "near" risk-less hedge strategy can also be found in the Samuelson and Samuelson/ Merton papers. Samuelson (1965) analyses the relationship between the expected return on the option (warrant), β , and the underlying stock, α , and argued that the difference "cannot become too large. If $\beta > \alpha$ […] hedging will stand to yield a sure-thing positive net capital gain (commissions and interest charges on capital aside!)"(p. 31). Samuelson/ Merton (1969) extend the earlier model and derive a "probability-cum-utility" function Q (see p. 19), which serves as a new probability measure (in today's terminology) to compute option prices. They show that under this new measure (or utility function), all securities earn the riskless rate; they explicitly write $\alpha_Q = \beta_Q = r$ to stress this point (see p. 26, equations 20 and 21 and the subsequent comments). Although Merton and Samuelson recognized the possibility of a (near) risk-less hedge and a risk-neutral valuation approach, they were not fully aware of the consequences of their findings.

[74] Black (1988) gives proper credit to Robert Merton: "Bob gave us that [arbitrage] argument. It should probably be called the Black-Merton-Scholes paper".

[75] Bernstein (1992) and Black (1989) provide interesting details about the birth of the Black-Scholes formula.

[76] The argument is derived from a funny analogy: He considers the mean (or fair) value of an asset as the *center of a circle, and* every point within the circle represents a possible future price. The radius describes the standard deviation. He then assumes that, as time elapses, the range of possible stock prices as represented by the area within the circle increases proportionally. This implies that the radius (i.e. the standard deviation) increases with the square root of time. A detailed analysis of Regnault's contribution is given in several papers by Jovanovic and Le Gall; see e.g. Jovanovic/ Le Gall (2001).

to Whelan (2002) who refers to a paper by Émile Dormoy published in 1873, French actuaries had a reasonable idea to price options well before Bachelier's thesis, although a clear mathematical framework was missing. Even Einstein in his Brownian motion paper (1905) did not quote Bachelier's thesis, but it is a generally accepted view that he did not know it. Distribution of knowledge was pretty slow at this time, particularly between different fields of research, and across different languages. And again, extensive references were simply not common in natural sciences (e.g. Einstein's paper contains a single reference to another author).

If Einstein did not know Bachelier's thesis, it is even less likely that Bronzin knew it; based on what we know from his other work (Bronzin 1906), his general mathematical interests were also quite apart from those of Bachelier. But after all, we do not know what Bronzin knew about other's work on option pricing, but the question is not so relevant either, because there are sufficiently many innovative elements in his treatise. It is also surprising that (almost) no references are found *on* his work, particularly in the German literature. Although it is generally claimed that Bachelier's thesis was lost until the Savage-Samuelson rediscovery, it was at least quoted since 1908 in several editions of a French actuarial textbook by Alfred Barriol (at least until 1925).

Bronzin's book had a similar recognition. As stated earlier, it was mentioned in Leitner's book about banking in Germany, published in four editions. And with Bronzin's more pragmatic pricing approach, it is difficult to understand why the seeds for another, more scientific understanding of option pricing did not develop, or the formulas did not get immediate practical attention. At least, Bronzin was not a doctoral candidate as Bachelier, but (apparently) a distinguished professor; moreover, the flourishing banking and insurance industry in Trieste should have had an active commercial interest in his research. While Poincaré's reservation on Bachelier's thesis is, at least, limited to his "queer" subject[77] and can, somehow, be understood from a purely academic point of view, it is more difficult to understand why a reviewer of Bronzin's book, in 1910, commented that "it can hardly be assumed that the results will attain a particularly practical value"…[78]. It however evidences Hans Bühlmann's and Shane Whelan's[79] claim that the contribution of actuaries to financial economics is generally underestimated (see Whelan 2002 for detailed references).

When comparing his contribution to Bachelier's thesis, then without any doubt, Bachelier was not only earlier, but his analysis is by far more rigorous from a mathematical point of view. Bronzin can not be credited having developed a new mathematical field, as Bachelier did with his theory on diffusions. Bronzin did *no* stochastic modeling, applied no stochastic calculus, derived no differential equations (except in the context of our equation 7), he was not interested in stochastic processes, and hence his notion of volatility has no time dimension. But except this, every element of modern option pricing is there! His contribution can be assessed as follows:

1. He noticed the unpredictability of speculative prices, and the need to use probability laws to price derivatives.
2. He recognized the informational role of market prices for pricing derivatives, and developed a theory relying on the current forward price of securities to price options. No expected values show up in the pricing formulas. His probability densities can be easily re-interpreted as risk-neutral pricing densities.

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[77] Poincaré was the main adviser of Bachelier's thesis; he writes in his report: «*Le sujet choisi par M. Bachelier s'éloigne un peu de ceux qui sont habituellement traités par nos candidats*»; Taqqu (2001), Appendix.

[78] Orig. text: „*Es ist kaum anzunehmen, daß die bezüglichen Resultate einen besonderen praktischen Wert erlangen können, wie ja übrigens auch der Verfasser selbst andeutet*". The last part of the sentence ("... which is also noticed by the author") is simply not true. The book review was published by an anonymous author in the *Monatshefte für Mathematik und Physik* (Volume 21; mit Unterstützung des Hohen K. K. Ministeriums für Kultus und Unterricht, Wien, Verlag des Mathematischen Seminars der Universität Wien).

[79] See Whelan (2002) for detailed references.

3. He understood the key role of arbitrage, although he is not very explicit about it; the derives the put-call parity condition, and uses a zero-profit condition to price forward contracts and options.
4. He develops a simplified procedure to find analytical solutions for option prices by exploiting a key relationship between their derivatives (with respect to their exercise prices) and the underlying pricing density. He also stresses the empirical advantages of this approach.
5. He extensively discusses how different distributional assumptions affect option prices. In particular, he shows how the normal law of error – which is the normal density function – can be used to price options, and how it is related to a binomial stock price distribution.
6. Besides of pricing simple calls and puts, he develops formula for chooser options and, more important, repeat-options.

His preference-free valuation equation (43) is closer to the Black-Scholes formula than anything else published before Black, Scholes and Merton. All this is a remarkable achievement, and it is done with a minimum of analytics. There are few things on the less elegant side: the discussion and the large systems of hedging conditions in the first part belongs to it, and some numerical procedures to solve for the repeat-option premiums also. But nevertheless, Bronzin's contribution is important, not only in historical retro-perspective. He definitively deserves his place in the history of option pricing, as other researchers as well. This was pointed out in a survey article by Girlich (2002). His introduction concludes our paper:

> "In the case of Louis Bachelier and his area of activity the dominant French point of view is the most natural thing in the world and every body is convinced by the results. The aim of the present paper is to add a

> few tesseras from other countries to the picture which is known about the birth of mathematical finance and its probabilistic environment."

We are happy having added another piece to this fascinating picture. Bachelier in his way was unique. But what is more important: he was not the only one who was working successfully on option pricing at the beginning of the 20^{th} century. So, the question remains why the results of Bachelier, Bronzin, and possibly other's yet to be re-discovered, did not get a broader acceptance? Why did their research not find immediate successors, academics that continued the way towards a practicable formula that made the pricing of options a subject of ongoing scientific research? Finding answers to these questions could help us to better understand the cultural background of financial mathematics, and would probably add an interesting chapter to the sociology of science. This would be a fascinating agenda of future research.

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